The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

1996 Problems

1. Let

$$f(x) = \sum_{k=1}^{n} \frac{k}{x-k}$$

Show that the set of real numbers x for which $f(x) \geq 1$ is a union of disjoint intervals with total length $\frac{1}{2}n(n+1)$.

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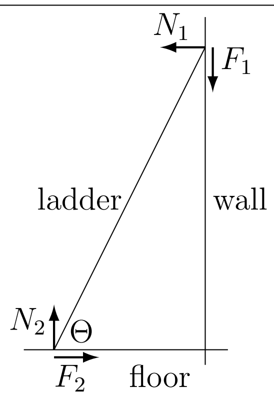
2. Given $f_1(x) = \dfrac{2x-1}{x+1}$ for $x \neq -1$, define $f_{n+1}(x) = f_1\big(f_n(x)\big)$ for $n = 1,\ 2,\ 3, \ldots$.

(a) Show that $f_{35}(x) = f_5(x)$.

(b) Determine $f_{28}(x)$.

3.



A ladder, modelled as a rigid rod of length $2a$ and weight W, leans against a vertical wall at an angle Θ as shown. The reaction forces on the ladder due to the wall and floor are given in terms of components (F_1, N_1) and (F_2, N_2), and the coefficients of friction due to the wall and floor are μ_1 and μ_2 respectively. When the ladder is at rest, $F_i \leq \mu_i N_i$ for $i = 1, 2$, with equality for either i only if the point of contact is on the verge of slipping.

(a) Show that if $\mu_1\mu_2 > 1$ then the ladder is in equilibrium at any angle Θ, but that N_1 and N_2 are not uniquely determined.

(b) Show that if $\mu_1\mu_2 < 1$ then there is a critical angle Θ_c such that if $\Theta < \Theta_c$ the ladder cannot be in equilibrium.

(c) Find Θ_c as a function of μ_1 and μ_2.

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4. A fair coin is tossed n times. What is the expected value of $|H - T|$, where H is the number of heads and T is the number of tails?

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5. Construct an example of a two-to-one function $f : [0,1] \longrightarrow \mathbb{R}$; that is, for each $y \in \mathbb{R}$, the set $\{x \in [0,1] : f(x) = y\}$ has exactly two elements or is empty. Show that no two-to-one function $f : [0,1] \longrightarrow \mathbb{R}$ is continuous on $[0,1]$.

∗6. Find all polynomials $p(x)$ such that $p(x)$ is an integer if and only if x is an integer.

∗7. Prove that the integers $\lfloor(\sqrt{2}+1)^n\rfloor$ are alternately even and odd. (Note that $\lfloor x\rfloor$ is the largest integer $\leq x$.)

∗8. Let $\boldsymbol{v}$, $\boldsymbol{u}$ and $\boldsymbol{w}$ be elements of the Euclidean plane $\mathbb{R}^2$. Show:

(a) $$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{v}\|+\|\boldsymbol{v}-\boldsymbol{w}\|+\|\boldsymbol{w}-\boldsymbol{u}\| &\leq 2\|\boldsymbol{u}\|+2\|\boldsymbol{v}\|+2\|\boldsymbol{w}\| \\ &\leq 3\|\boldsymbol{u}+\boldsymbol{v}\|+3\|\boldsymbol{v}+\boldsymbol{w}\|+3\|\boldsymbol{w}+\boldsymbol{u}\|,\end{aligned}$$

(b) $\|\boldsymbol{u}\|\,\|\boldsymbol{v}-\boldsymbol{w}\| \leq \|\boldsymbol{v}\|\,\|\boldsymbol{w}-\boldsymbol{u}\|+\|\boldsymbol{w}\|\,\|\boldsymbol{u}-\boldsymbol{v}\|$.

∗9. Define a sequence of positive integers by:

$$a(1)=a(2)=1,\ a(n)=a\big(a(n-1)\big)+a\big(n-a(n-1)\big) \text{ for } n\geq 3.$$

Prove that $a(2^n)=2^{n-1}$ for all positive integers n.

∗10. Let $f(x)$ be a continuously differentiable function of a real variable x (i.e. f is differentiable on $\mathbb{R}$ and f' is continuous on $\mathbb{R}$.) Let

$$\lim_{x\to\infty}\Big(\big(f'(x)\big)^2+\big(f(x)\big)^3\Big)=0.$$

Show that $\lim\limits_{x\to\infty} f(x)=0$ and $\lim\limits_{x\to\infty} f'(x)=0$.
