1. Let 

\[ f(x) = \sum_{k=1}^{n} \frac{k}{x-k} \]

Show that the set of real numbers \( x \) for which \( f(x) \geq 1 \) is a union of disjoint intervals with total length \( \frac{1}{2}n(n+1) \).

2. Given \( f_1(x) = \frac{2x-1}{x+1} \) for \( x \neq -1 \), define \( f_{n+1}(x) = f_1(f_n(x)) \) for \( n = 1, 2, 3, \ldots \).
   (a) Show that \( f_{35}(x) = f_5(x) \).
   (b) Determine \( f_{28}(x) \).

3. A ladder, modelled as a rigid rod of length \( 2a \) and weight \( W \), leans against a vertical wall at an angle \( \Theta \) as shown. The reaction forces on the ladder due to the wall and floor are given in terms of components \( (F_1, N_1) \) and \( (F_2, N_2) \), and the coefficients of friction due to the wall and floor are \( \mu_1 \) and \( \mu_2 \) respectively.

   When the ladder is at rest, \( F_i \leq \mu_i N_i \) for \( i = 1, 2 \), with equality for either \( i \) only if the point of contact is on the verge of slipping.
   (a) Show that if \( \mu_1 \mu_2 > 1 \) then the ladder is in equilibrium at any angle \( \Theta \), but that \( N_1 \) and \( N_2 \) are not uniquely determined.
   (b) Show that if \( \mu_1 \mu_2 < 1 \) then there is a critical angle \( \Theta_c \) such that if \( \Theta < \Theta_c \) the ladder cannot be in equilibrium.
   (c) Find \( \Theta_c \) as a function of \( \mu_1 \) and \( \mu_2 \).

4. A fair coin is tossed \( n \) times. What is the expected value of \( |H - T| \), where \( H \) is the number of heads and \( T \) is the number of tails?

5. Construct an example of a two-to-one function \( f : [0,1] \rightarrow \mathbb{R} \); that is, for each \( y \in \mathbb{R} \), the set \( \{ x \in [0,1] : f(x) = y \} \) has exactly two elements or is empty. Show that no two-to-one function \( f : [0,1] \rightarrow \mathbb{R} \) is continuous on \([0,1]\).
**6.** Find all polynomials $p(x)$ such that $p(x)$ is an integer if and only if $x$ is an integer.

**7.** Prove that the integers $\lfloor (\sqrt{2} + 1)^n \rfloor$ are alternately even and odd. (Note that $\lfloor x \rfloor$ is the largest integer $\leq x$.)

**8.** Let $v, u$ and $w$ be elements of the Euclidean plane $\mathbb{R}^2$. Show:

\begin{align*}
(a) \quad & \|u - v\| + \|v - w\| + \|w - u\| \leq 2\|u\| + 2\|v\| + 2\|w\| \\
& \leq 3\|u + v\| + 3\|v + w\| + 3\|w + u\|,

(b) \quad & \|u\| \|v - w\| \leq \|v\| \|w - u\| + \|w\| \|u - v\|.
\end{align*}

**9.** Define a sequence of positive integers by:

$$a(1) = a(2) = 1, \ a(n) = a(a(n - 1)) + a(n - a(n - 1)) \text{ for } n \geq 3.$$ 

Prove that $a(2^n) = 2^{n-1}$ for all positive integers $n$.

**10.** Let $f(x)$ be a continuously differentiable function of a real variable $x$ (i.e. $f$ is differentiable on $\mathbb{R}$ and $f'$ is continuous on $\mathbb{R}$.) Let

$$\lim_{x \to \infty} \left( (f'(x))^2 + (f(x))^3 \right) = 0.$$ 

Show that $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to \infty} f'(x) = 0$. 