The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

1999 Problems

1. A continuous function. A real function is continuous at a point a. Prove or disprove that it is continuous on a neighbourhood of a.

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2. Equal polynomials. Let p and q be complex polynomials, not both constant. Prove or disprove:

$$\text{If } p(z) = 0 \iff q(z) = 0 \text{ and } p(z) = 1 \iff q(z) = 1, \text{ then } p = q.$$

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3. A functional equation. Let C be a real number. Find all differentiable functions f satisfying

$$f(x + y) = Cf(x)f(y).$$

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4. Complex numbers. Prove that $|1 + z| \leq |1 + z|^2 + |z|$ for all complex numbers z.

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5. A mean value theorem. Let $f(x)$ be continuous on $[a, b]$, differentiable on (a, b) and satisfy

$$\big(f(a)\big)^2 - \big(f(b)\big)^2 = b^2 - a^2.$$

Show that there exists $c \in (a, b)$ such that $f'(c) \cdot f(c) + c = 0$.

***6. The average mean.** Show that the mean of a finite set of numbers is the mean of the means of all non-empty subsets of the set.

***7. Subspaces.** Let V be a vector space. For any subset S of V, let $\mathrm{Diff}(S)$ be the set of differences $\{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v}, \boldsymbol{u} \in S\}$. Show that if A and B are subsets of V such that $V = A \cup B$, then either $\mathrm{Diff}(A) \supseteq B$ or $\mathrm{Diff}(B) \supseteq A$. Show that if also $A \cap B \neq \emptyset$ then $\mathrm{Diff}(A) = V$ or $\mathrm{Diff}(B) = V$.

***8. Inscribed figures.** Find the possible areas of equilateral triangles inscribed in a unit square, that is, each vertex lies on a side or corner of the square.

***9. Random triangles.** Three numbers are chosen at random without replacement from the first n natural numbers. What is the probability that they can be the sides of a triangle?

***10. Balancing weights.** Suppose that one has a number of blocks of integer weights which average 2 units, and such that the heaviest block is not heavier than the rest of the blocks combined. With the exception of the case of an odd number of blocks of weight 2, is it possible to separate the blocks into two sets of equal weight as follows?

> Take a two-pan balance and put the heaviest weight on one pan. Then proceeding step-by-step place the next heaviest block on the pan which which is lighter, choosing either if the pans are balanced, finally placing the lightest weight to make the pans balance.
