The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2001 Problems with Solutions

1. A recursively defined function. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2 + 1 \text{ for } -1 < x \leq 1 \text{ and } f(x+2) = 1/f(x) \text{ for all } x \in \mathbb{R}.$$

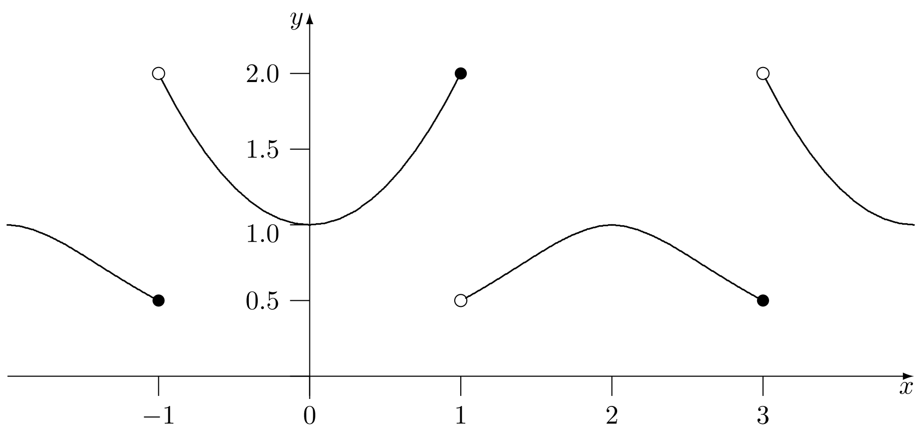
Sketch the graph of $f(x)$, find the period, and find an explicit formula for $f(x)$ in the interval $(1, 3]$.

Solution. Firstly,

$$f(x+4) = f(x+2+2) = \frac{1}{f(x+2)} = \frac{1}{1/f(x)} = f(x).$$

Hence f is periodic, with period 4.
Now for $x \in (1, 3]$,

$$\begin{aligned} f(x) = f(x-2+2) &= \frac{1}{f(x-2)} \\ &= \frac{1}{(x-2)^2+1}. \end{aligned}$$



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2. Suppose $\ddot{x} - x \leq 0$, $x(0) = 1$, $\dot{x}(0) = 1$.

Prove that $x(t) \leq e^t$ for all $t \in [0, \infty)$.

Is the converse true?

[$\dot{x}$ means $\dfrac{dx}{dt}$ and $\ddot{x}$ means $\dfrac{d^2x}{dt^2}$.]

Solution. We infer from the existence of $\ddot{x}$ on $[0, \infty)$ that $\dot{x}$ and therefore x also are continuous on $[0, \infty)$.

$$\begin{aligned}\frac{d}{dt}\Big((\dot{x} - x)\, e^t\Big) &= (\ddot{x} - x)\, e^t + (\dot{x} - x)\, e^t \\ &= (\ddot{x} - x)\, e^t \\ &\leq 0.\end{aligned}$$

So, $(\dot{x} - x)\, e^t$ is a non-increasing function on $[0, \infty)$. Hence, for $t \geq 0$,

$$\begin{aligned}&(\dot{x} - x)\, e^t \leq \big(\dot{x}(0) - x(0)\big)\, e^0 = 0 \\ \implies\ & \dot{x} - x \leq 0 \\ \implies\ & (\dot{x} - x)\, e^{-t} \leq 0 \\ \implies\ & \frac{d}{dt}\Big(xe^{-t}\Big) \leq 0 \\ \implies\ & xe^{-t} \leq x(0)\, e^{-0} = 1 \\ \implies\ & x(t) \leq e^t.\end{aligned}$$

Alternatively, $\ddot{x} - x \leq 0$ implies $\ddot{x} - x = f(t)$ for some $f(t) \leq 0$. Taking Laplace transforms, we have:

$$\begin{aligned}&\big(s^2 X(s) - s - 1\big) - X(s) = F(s) \\ \implies\ & X(s) = \frac{1}{s-1} + \frac{F(s)}{s^2 - 1}.\end{aligned}$$

Hence $x(t) = e^t + \int_0^t f(u) \sinh(t - u)\, du$, by the Convolution Theorem.
For $0 \leq u \leq t$, $\sinh(t - u) \geq 0$ and $f(u) \leq 0$. Hence the integrand is ≤ 0, so that the integral is ≤ 0. Thus, $x \leq e^t$.

The converse is false: let $x(t) = e^t \cos t$. Then

$$\ddot{x} - x = -e^t(2 \sin t + \cos t).$$

So we have $x(0) = \dot{x}(0) = 1$ and $x(t) \leq e^t$ but $\ddot{x}(\pi) - x(\pi) = e^\pi \not\leq 0$.

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3. Three-dimensional geometry. L and M are skew lines in space such that L is parallel to a line perpendicular to M. A line segment PQ of fixed length moves so that P is on L and Q is on M. What is the locus of the mid-point of PQ?

Solution. Let c be the distance between L and M and set up the coordinates so that the origin is the midpoint of the shortest line segment between L and M with the x-axis above L and the y-axis below M. Then the coordinates of P and Q are of the form $P(a, 0, -c/2)$ and $Q(0, b, c/2)$; so the midpoint is $R(a/2, b/2, 0)$. Let d be the distance between P and Q. Then $a^2 + b^2 = d^2 - c^2$; so the locus of midpoints is the circle

$$x^2 + y^2 = \frac{d^2 - c^2}{4}.$$

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4. **Constructing the Great Pyramid.** A recent theory to explain how the 2.5 tonne stone blocks forming the Great Pyramid of Egypt were moved suggested that cradles, shaped as arcs of a circle were attached to the blocks, effectively making them cylinders.

Long ropes were then coiled round the cylinder. As the slaves pulled on the ropes, the ropes unwound and the blocks were rolled up ramps believed to have a slope of one-in-four.

In a modern test simulating the conditions, it was found that 3 men were able to roll the cradled stone on level ground while 20 men were able to roll the stone up the ramp.

How many men would be required to just prevent the block and cradle rolling down the ramp?

Estimate the average pull exerted by each man to roll the stone up the ramp, and decide whether the suggested method is feasible.

Solution. Since any motion will be slow, it is reasonable to ignore acceleration, so the following simplified model considers static equilibrium of forces.
Let M kg weight $= 9.8$ Newtons be the pull exerted by one man. Let F be the frictional force, and W the weight force. Then on level ground

$$F = 3M.$$

When pulling up the slope, $20M$ balances the component of the weight force parallel to the plane and the friction force.

$$\begin{aligned} W\sin\alpha + F &= 20M \\ \implies M &= \frac{W\sin\alpha}{17} \\ &\approx \frac{2500}{6\times 17}\,\text{kg weight} \\ &\approx 37\,\text{kg weight} \\ &\approx 360\,\text{Newtons.} \end{aligned}$$

Let n be the number of men required to hold against motion down the plane. Then nM and the friction force balances the weight components.

$$nM + F = W\sin\alpha \implies n = 14.$$

It does seem feasible that a man can exert a pull of 37 kg weight.

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5. **Random Queen Moves.** If two queens are randomly placed on distinct squares of an ordinary chessboard, what is the probability that they attack each other?

Solution. Let (a, b) and (x, y) be the squares occupied by the two queens. The queens attack each other if and only if:

(i) $a = x$, or

(ii) $b = y$, or

(iii) $a + b = x + y$, or

(iv) $a - b = x - y$,

and only one of these conditions can hold.
For each of (i) or (ii) there are $8 \times 8 \times 7 = 448$ possible attacking positions, and for each of (iii) and (iv) there are

$$2\sum_{r=1}^{7} r(r-1) + 8(8-1) = 2 \times 112 + 56 = 280 \,\text{positions.}$$

Hence the total number of pairs of attacking positions is 1456 out of a total number 64×63 of pairs of positions.
So the required probability is s

$$\frac{1456}{64} \times 63 = \frac{13}{36}.$$

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6. Two nice limits. Let A_n and G_n be respectively the arithmetic and geometric mean of the n positive integers $n+1,\ n+2, \ldots, n+n$. Find the limits as $n \to \infty$ of A_n/n and G_n/n. [*Hint.* For G_n think Riemann integral.]

Solution. [G_n/n **limit by Justin Foo**]
Firstly,

$$\begin{aligned} A_n &= \frac{1}{n}\Big((2n/2)(2n+1) - (n/2)(n+1)\Big) \\ &= \frac{3n+1}{2} \\ \therefore \frac{A_n}{n} &= \frac{3+1/n}{2} \to \frac{3}{2} \text{ as } n \to \infty \end{aligned}$$

Now, $G_n^n = (n+1)(n+2)\ldots(2n)$. So

$$G_n = \exp\Big(\frac{1}{n}\sum_{i=1}^{n} \ln(n+i)\Big).$$

The expression $\sum_{i=1}^{n} \ln(n+i)$ is a Riemann sum for $\ln x$ over the interval $[n, 2n]$ so in the limit, as $n \to \infty$,

$$\begin{aligned} \frac{1}{n}\left(\sum_{i=1}^{n} \ln(n+i)\right) \to \int_n^{2n} \frac{\ln x}{n}\, dx &= \frac{1}{n}\Big[x(\ln x - 1)\Big]_n^{2n} \\ &= \ln n + 2\ln 2 - 1 = \ln\Big(\frac{4n}{e}\Big). \end{aligned}$$

Thus $\sum_{i=1}^{n} \ln(n+i)/n = \ln(4n/e) + k$ where $k \to 0$ as $n \to \infty$. So

$$\begin{aligned} \lim_{n\to\infty} \frac{G_n}{n} &= \lim_{n\to\infty} \frac{\exp\big(\ln(4n/e) + k\big)}{n} \\ &= \lim_{n\to\infty} \frac{e^k 4n/e}{n} \\ &= \frac{4}{e} \text{ since } e^k \to 1 \text{ as } n \to \infty. \end{aligned}$$

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7. Linear Independence. Let V be the vector space of real valued functions on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(a) Prove that the four functions $\sin x$, $\cos x$, $\tan x$ and $\sec x$ are linearly independent.

(b) Let W be the span of the four functions $\sin x$, $\cos x$, $\tan x$ and $\sec x$. Let $T : W \longrightarrow V$ be the linear transformation defined on the basis elements by $T(\sin x) = \sin^2 x$, $T(\cos x) = \cos^2 x$, $T(\tan x) = \tan^2 x$ and $T(\sec x) = \sec^2 x$. Find a basis for the kernel of T.

Solution. [**Part (b) by Justin Foo**]

(a) If $a \sin x + b \cos x + c \tan x + d \sec x = 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then (with $x = 0$), $b = -d$; so, (with $x \to \pi/2$), $a = 0$ and $c + d = 0$. Hence, (with $x = \pi/4$), $b/\sqrt{2} + c + d/\sqrt{2} = 0$. Hence $a = b = c = d = 0$. Thus, since the only linear combination of $\sin x$, $\cos x$, $\tan x$ and $\sec x$ that is identically zero, is the trivial one, $\sin x$, $\cos x$, $\tan x$ and $\sec x$ are linearly independent.

(b) Note that $\sec^2 x = 1 + \tan^2 x = \sin^2 x + \cos^2 x + \tan^2 x$. Hence $\sin x + \cos x + \tan x - \sec x$ is in the kernel of T. On the other hand, using in turn $x = \pi/3$, $x = \pi/4$ and $x = \pi/6$, we see that $\{\sin^2 x, \cos^2 x, \tan^2 x\}$ is linearly independent. Hence $\ker T$ has dimension 1 and basis $\{\sin x + \cos x + \tan x - \sec x\}$.

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8. A continuous non-differentiable function. Define $f : (0, 1) \to (0, 1)$ as follows:

For each $x \in (0, 1)$, write x in decimal notation $0.x_1x_2\dots$ where the $x_i \in \{0, 1, \dots 9\}$ and the representation does not end in all 9s. Let $f(x) = y$ be the decimal obtained from x by changing each 1 to 2 and each 2 to 1.

Show that f is continuous on $(0, 1)$, $\int_0^1 f(x)\,dx = 1/2$ and f is not differentiable anywhere on $(0, 1)$.

Solution. Let $a = 0.a_1a_2\dots \in (0, 1)$.
For any m and $0.a_1a_2\dots a_m \leq x < a_1a_2\dots a_m999\dots$, $f(x)$ agrees with $f(a)$ in the first m digits; so $|f(x) - f(a)| \leq 10^{-m}$. Therefore f is continuous from the right at a. The same inference is valid from the left, unless $a = 0.a_1a_2\dots a_n$, but then for sufficiently large m we can apply the same argument.
Now note that the integral exists since f is continuous and bounded. Partition the interval into 10^n equal parts and use the values of f at the left endpoints. The Riemann Sum is a re-ordered sum for $\int_0^1 x\,dx = 1/2$.
Assume f is differentiable at some $a = 0.a_1a_2\dots$. Let d be any digit that appears infinitely many times. Let $x_{i,n}$ be the number obtained by replacing the n^{th} occurrence of d by i, $i = 1, 2$. Thus $x_{i,n} \to a$ as $n \to \infty$. The quotient

$$\frac{f(x_{i,n}) - f(a)}{x_{i,n} - a}$$

is constant. The limits of the quotients must be the same for both choices of i. But one has denominator 1 and another -1, a contradiction. Thus, f is not differentiable for any $a \in (0, 1)$.

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9. Similar matrices. Real $n \times n$ matrices A and B are called **real similar** if there is an invertible real matrix S such that $S^{-1}AS = B$ and **complex similar** if there is an invertible complex matrix T such that $T^{-1}AT = B$. Prove that if A and B are complex similar then they are real similar.

> **Solution.** Let $T^{-1}AT = B$ for a complex matrix T. There are real matrices P and Q such that $T = P + iQ$.
> Then $A(P + iQ) = (P + iQ)B$, so $AP = PB$ and $AQ = QB$. If P or Q are non-singular we are done. Suppose they are both singular.
> The polynomial $p(x) = \det(P + xQ)$ is not the zero polynomial since $p(i) = \det T \neq 0$. Hence there is a real number r such that $p(r) \neq 0$. Hence $P + rQ$ is a real nonsingular matrix such that $A(P + rQ) = (P + rQ)B$.

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10. An inequality.

(a) Let x and y be positive real numbers. Prove that $x^y + y^x \geq 1$.

(b) Let x, y and z be positive real numbers. Prove that $x^y + y^z + z^x \geq 1$.

> **Solution.**
>
> (a) First note that if $f(x) = x^x$ then $f'(x) = x^x(\log x + 1)$ so f has a unique turning point at $x = 1/e$ and $f(1/e) = e^{-1/e} = A$ say ($A \approx 0.69$). Since $f(1) = 1$ and $\lim_{x \to 0^+} = 1$, this is a minimum.
> The inequality is certainly true if x or $y \geq 1$, so we may assume $0 < y < x < 1$. Put $y = kx$ for some $k < 1$. Let $F(x) = x^y + y^x = x^{kx} + (kx)^x = (x^x)^k + k^x x^x$.
> Since $k < 1$ and $x < 1$, $k^x > k$ so $F(x) \geq A^k + kA$. Now $A^k + kA$ (as a function of k) has a unique minimum at $k = 1 - e$ and is increasing for $0 < k < 1$. Hence the minimum value of $A^k + kA$ is 1, when $k = 0$. Hence $F(x) > 1$.
>
> (b) We can assume $x \leq y \leq z$; so $S \geq x^y + y^z \geq x^y + y^x \geq 1$.

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11. An abstract operation. Let $*$ be an operation on a set S, i.e. for all a and $b \in S$, $a * b$ is an element of S. Call $e \in S$ a **near identity** if $e * e = e$ and $e * x = x * e = x$ fails for at most one $x \in S$.

(a) Show that if $*$ is associative then S has at most two near identities.

(b) Give an example of a finite set S and an operation $*$ such that $*$ is associative and S has two near identities.

(c) Give an example of an infinite set S and an operation $*$ such that $*$ is associative and S has two near identities.

> **Solution.** [**Part (b), (c) examples by Justin Foo**]
>
> (a) Say that e works for x if $e^2 = e$ and $ex = xe = x$. If e and f are distinct near identities then it cannot be that each works for the other, since in that case, $e = fe = ef = f$.
> Assume that S has 3 distinct near identities, e, f and g. Then each must work for at least one of the other two, e works for f. Then f must work for g and g for e. But then $e = eg = e(gf) = (eg)f = ef = f$.

(b) For a finite example, take

$$S = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

under ordinary matrix multiplication.

(c) For an infinite example, take

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} x & y \\ x & y \end{bmatrix} : x, y \in R \right\}$$

under ordinary matrix multiplication.