The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2002 Problems with Solutions

---

1. Matrix Squares. Let A be an $n \times n$ matrix such that $A^{2m+1} = I_n$ for some natural number m.

Show that A is the square of an invertible matrix.

> **Solution.** Since $A^{2m}A = I_n$, A is invertible, and has inverse A^{2m}.
> Hence A^{m+1} is invertible and $(A^{m+1})^2 = A^{2m+2} = A^{2m+1}A = A$.

---

2. Conic Tangents. Let the parabola $x = y^2$ and the circle $x^2 + y^2 = 1$ meet at P above the x-axis and Q below the x-axis. Tangents common to the circle and parabola meet the parabola at X below the x-axis and Y above the x-axis.

Prove that XP and YQ are tangent to the circle.

Solution. [**By Michael Pauley, 1st year, UWA**]
The coordinates of Q are $(g, -\sqrt{g})$, where

$$g = \frac{\sqrt{5}-1}{2}.$$

By similar triangles, the coordinates of Y are $(c, \sqrt{c})$, where $c = 2 + \sqrt{5}$.
The tangent to the circle at Q has slope $\sqrt{g}$, so passes through $(c, \sqrt{c}) = Y$.
Alternatively, we have $P(a, \sqrt{a})$ and $Q(a, -\sqrt{a})$, where $a = (\sqrt{5}-1)/2$. The tangent at the point (t^2, t) on the parabola has equation $2ty = x + t^2$ so its distance from the origin is $t^2(4t^2+1)^{-\frac{1}{2}}$. Setting this distance equal to 1 gives $X = (c, -\sqrt{c})$ and $Y = (c, \sqrt{c})$ where $c = 2 + \sqrt{5}$. Hence the dot product $\overrightarrow{OP} \cdot \overrightarrow{OX} = 1$ and so

$$\overrightarrow{OP} \cdot \overrightarrow{PX} = \overrightarrow{OP} \cdot (\overrightarrow{OX} - \overrightarrow{OP}) = 1 - 1 = 0.$$

Thus PX is tangent to the circle and similarly, so is YQ.

---

3. Who Won?

Alice: Did you hear about the penalty kick shootout between Turkey and Spain?
Bob: I heard only that Turkey shot first. What was the score? Who won?
Alice: The score was interesting. If I told you the score, you could tell me who won.
Bob: Well, then I know who won even without knowing the score!

Who won?

[*Explanation:* A penalty kick shootout in soccer works as follows. There are five rounds of the form: Team A shoots; then team B shoots. However, if at any point the score is such that a team is guaranteed to win, the shootout ends. If they are tied after five rounds, then additional rounds are played until there is a winner. A score is reported as m-to-n where $m > n$ (regardless of who shot first).

Solution. There are two ways to solve this. All the submitted solutions did it the brute force way, showing that the only scores satisfying the conditions of the problem are 4–1 to Turkey and 5–3 to Turkey.

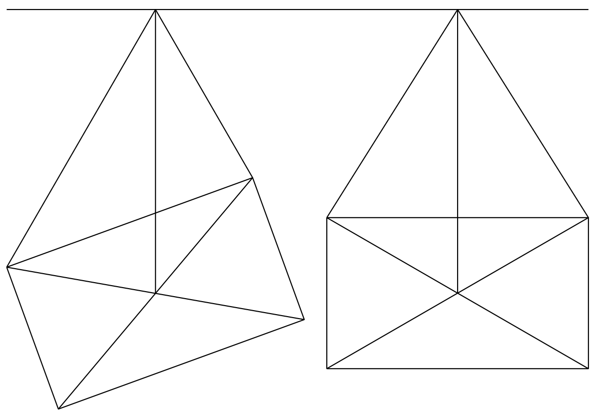
More elegantly, suppose there is a sequence S_1 of shots by which Spain could win. If Turkey shot last in this sequence, it would make no difference if the winner Spain took another shot which missed. Hence we may suppose that both sides shot the same number of times. Therefore Turkey could have won by the same score by the sequence S_2 in which Turkey scored [missed] on shot i if and only if Spain scored [missed] on shot i in S_1. Hence Spain could not have won by a score that Turkey could not win by.

But we know that Turkey could win by a score that Spain could not win by. Therefore Turkey won.

---

4. Hanging a Picture. Suppose a rectangular picture of length a and height b is suspended by a cord of length ℓ attached to its two upper corners, passing over a hook in the wall. Assume that the weight of the cord, the friction between the cord and the hook, and between the picture and the wall are negligible.

(a) Show that there is a critical length L such that if $\ell > L$ then the picture will hang in stable equilibrium in a horizontal position, but if $\ell < L$ then the horizontal picture, if slightly disturbed, will slide into a stable but slanting equilibrium position with its centre vertically below the hook.



(b) Show that at a position of tilted equilibrium, the angle at the hook between the two pieces of cord equals the small angle between the diagonals of the picture.

(c) Find L in terms of a and b.

(d) As the cord length is varied, the equilibrium position of the picture moves relative to the hook. Find the locus of the hook relative to the equilibrium position of the picture.

Solution. [**By combining solutions of Michael Pauley, 1st year UWA, and Daniel Stefannelli, 2nd Year UWA**]
(a) and (b) Call the top corners A and B, the hook H, and the centre of the picture O. Let the small angle between the diagonals be θ, so the large angle is $180° - \theta$.
When the picture is not moving, the vertical through H passes through O and the tensions in the both parts of the string are equal, so $\angle AHO = \angle BHO = \alpha$, say. By the Sine Rule in triangles AHO and BHO, $\sin HAO = \sin HBO$; so either these angles are equal, or have sum $180°$. Hence either $2\alpha < \theta$ and the picture is in stable equilibrium in the horizontal position, or $2\alpha = \theta$ and the picture is in stable equilibrium in one of the two slanted positions.
(c) By the geometry of the configuration, for slanted equilibrium, the critical length L satisfies $L \sin \alpha = a$ and

$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}.$$

So,

$$L = \frac{a}{b}\sqrt{a^2 + b^2}.$$

(d) In slanted equilibrium, the angle at H is constant, so as the length increases from a to L, the hook moves relative to the picture in an arc of a circle passing through A, B and O. As the length exceeds L, it moves up along the vertical through O.

---

5. Time flies. The hands of a clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly.

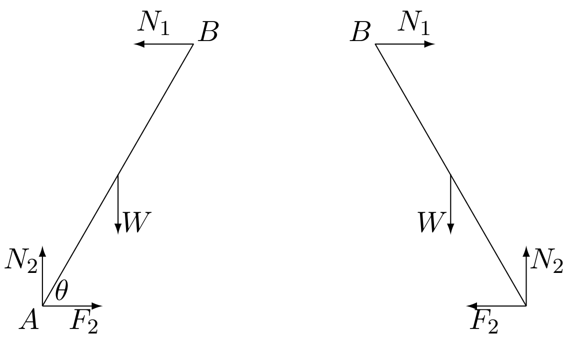
Solution. Let $\overrightarrow{OA}$ be the big hand and $\overrightarrow{OB}$ the little hand. We can think of $\overrightarrow{OA}$ as fixed and $\overrightarrow{OB}$ rotating at constant speed. Let $\underset{\sim}{v}$ be the velocity vector of B under this assumption. The rate of change of distance AB is the component of $\underset{\sim}{v}$ in direction $\overrightarrow{AB}$. Since $\underset{\sim}{v} \perp \overrightarrow{OB}$ and $|\underset{\sim}{v}|$ is constant, this component is maximal when $\angle OBA$ is a right angle, i.e. distance $AB = \sqrt{4^2 - 3^2} = \sqrt{7}$.

---

6. House of cards. When building a house of cards, the first step is to arrange two cards in an inverted V. How rough must the table be in order that the cards stay in place?

[*Hint.* Roughness is measured by the coefficient of friction, defined as follows: a contact force between two objects may be resolved into a friction component F and a normal component N. The coefficient of friction is a constant μ such that $F = \mu N$ when one of the objects is at the point of slipping.]

Solution.



Let A, B, θ as indicated in the diagram, and let a be the length of the card AB.

By symmetry the action of each card on the other at B must be horizontal. (For if not, an upward component would have to equal a downward component.)
Balance of forces gives: $N_2 = W$, $N_1 = F_2$.
Balance of moments about A gives:

$$W \cdot a \cdot \sin\theta = N_1 \cdot 2a \cdot \cos\theta$$
$$\therefore F_2 = \frac{W \tan\theta}{2}$$

Now the Friction Law gives: $F_2 \le \mu N_2$ from which we deduce

$$\mu \ge \tfrac{1}{2}\tan\theta.$$

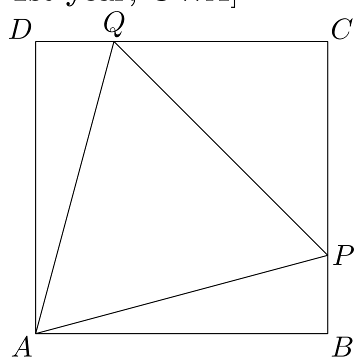
---

7. Equilateral triangle inscribed in a rectangle. On the sides BC and CD of rectangle $ABCD$, points P and Q are drawn such that APQ is an equilateral trangle.

(a) Prove that Area($\triangle AQD$) + Area($\triangle APB$) = Area($\triangle PCQ$).

(b) Determine the possibilities for the ratio $BC : AB$.

Solution. [**By Alix Nulsen, 1st year, UWA**]



Assume the triangle has side length 1, since this makes no difference to the problem or its solution.

(a) Let $\angle BAP = \theta$, X = Area($\triangle APB$), Y = Area($\triangle AQD$), and Z = Area($\triangle PCQ$). Then

$$X = \tfrac{1}{2}\cos\theta\sin\theta = \tfrac{1}{4}\sin(2\theta).$$

Similarly,

$$Y = \tfrac{1}{4}\sin(\pi/3 - 2\theta) \text{ and}$$
$$Z = \tfrac{1}{4}\sin(\pi/3 + 2\theta).$$

Hence

$$\begin{aligned} X + Y &= \tfrac{1}{4}\big(\sin(2\theta) + \sin(\pi/3 - 2\theta)\big) \\ &= \tfrac{1}{4}\big(\sin(2\theta) + \sin(\pi/3)\cos(2\theta) - \tfrac{1}{2}\sin(2\theta)\big) \\ &= \tfrac{1}{4}\big(\tfrac{1}{2}\sin(2\theta) + \sin(\pi/3)\cos(2\theta)\big) \\ &= \tfrac{1}{4}\sin(\pi/3 + 2\theta) \\ &= \tfrac{1}{4}\sin\big(2(\pi/6 + \theta)\big) \\ &= Z \end{aligned}$$

Hence Area($\triangle AQD$) + Area($\triangle APB$) = Area($\triangle PCQ$).

(b) As θ varies continuously from its minimum 0 to its maximum $\pi/6$, the ratio BC/AB varies from its maximum $1/(\sqrt{3}/2)$ to its minimum $\sqrt{3}/2$. Hence

$$\sqrt{3}/2 \leq BC : AB \leq 2/\sqrt{3}.$$

---

8. A random walk. A $3 \times 3 \times 3$ cube is divided into 27 unit cubelets. There is a passage allowed between any pair of adjacent cubelets (i.e. those which share a common face).

A mouse is placed in one of the corner cubelets and a block of cheese in the centre cubelet. The mouse wanders at random between adjacent cubelets looking for the cheese. On average, how many moves will she have to make to find the cheese?

Solution. To reach the centre, the mouse will have to move from the corner to the middle of an edge and then to the middle of a face and finally the centre. Suppose C, E and F are the expected number of steps the mouse needs to take to reach the centre from a corner, middle of an edge and middle of a face respectively.
Note that from a corner, the mouse must be at the middle of an edge at the next step and hence $C = 1 + E$ or $E = C - 1$. From the middle of an edge, half the time the mouse will go back to the corner needing an additional C steps, on an average, to reach centre, while half the time she will move to the middle of a face, needing an additional F steps on an average. Thus we have $E = 1 + \frac{1}{2}(C + F)$ and using the relationship between E and C we get $C = 4 + F$ or $F = C - 4$.
From the middle of a face at the next step the mouse may reach the centre one fifth of the time while four fifths of the time she will go back to middle of an edge needing an extra E more steps on average. Thus $F = 1 + \frac{4}{5}E = \frac{1}{5} + \frac{4}{5}C$. Substituting $C - 4$ for F and solving for C, we get $C = 21$.
Alternatively, let $p(n)$ be the probability of reaching centre from the middle of an edge in n steps. Obviously, $p(1) = 0$ and $p(2) = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$. For $n > 2$, in the first two moves, the mouse either will have to go to a corner and then back to the centre of an edge (for which the probability is $\frac{1}{2}$) or will have to go to centre of a face and then come back to a centre of the edge (which has a probability of $\frac{1}{2} \cdot \frac{4}{5} = \frac{4}{10}$ and then reach the centre in $n - 2$ steps and so we have the recursive relationship $p(n) = (\frac{1}{2} + \frac{4}{5})p(n-2) = \frac{9}{10}p(n-2)$, from which we have $p(n) = (\frac{9}{10})^{(n-2)/2} \cdot \frac{1}{10}$ for even n and 0 for odd n. Thus the expected number of steps needed to reach the centre from the middle of an edge is $\sum_{m=1}^{\infty} 2m(\frac{9}{10})^{m-1} \cdot \frac{1}{10} = 20$, on comparing the sum with the binomial expansion of $(1-p)^{-2}$ for $p = \frac{9}{10}$. Now add the extra step to reach the edge from the corner, and we find the expected number is $20 + 1 = 21$.

---

9. Infinitely many equations.

(a) For each positive integer m, let $E(m)$ be the equation in two unknowns $(x-m) - y_m^2 = 0$, so we have a system of equations in unknowns x, y_1, $y_2, \ldots$. Show that for every m, the first m equations $E(1)$, $E(2)$, $\ldots E(m)$ of this system have a real solution but the whole system does not.

(b) For each complex number c, let $E(c)$ be the equation in two unknowns $(z-c)y_c - 1 = 0$. Show that every **proper** subset of the system has a complex solution, but the whole system does not.

Solution. [**By Michael Pauley, 1st year, UWA**]

(a) Given a finite subsystem, let y_m be the first unknown that does not appear in the subsystem. Then

$$x = m,\ y_1 = (m-1)^{\frac{1}{2}},\ y_2 = (m-2)^{\frac{1}{2}}, \dots,\ y_{m-1} = 1$$

contains a solution to the finite subsystem. However, the whole system does not, because if there were a solution with $x = r$, let $m > r$ be an integer. Then there is no y_m satisfying $(x-m) - y_m^2 = 0$, a contradiction.

(b) Given a proper subsystem P, let $a \in \mathbb{C}$ be a complex number such that $(z-a)y_a - 1$ does not appear in P. A solution of P is $z = a,\ y_c = (a-c)^{-1}$. But the entire system has no solution, because if there were a solution with $z = u \in \mathbb{C}$, then there is no y_u such that $(z-u)y_u - 1 = 0$, a contradiction.

---

10. Squares in a sequence. Let $a_1 = 1$ and $a_{n+1} = a_n + \lfloor\sqrt{a_n}\rfloor$ for $n = 1, 2, \dots$. Show that a_n is a square if and only if $n = 2^k + k - 2$ for some positive integer k.

Solution. We will show by induction on k that $a_{n_k} = (2^{k-1})^2$ for $n_k = 2^k + k - 2$ and a_m is not square for $n_{k-1} < m < n_k$.
This is true for $k = 1$. Assume it is true for some k. By induction on i,

$$a_{n+2i+1} = (2^{k-1} + i)^2 + 2^{k-1} - i,\ \text{for } 0 \le i \le 2^{k-1} \quad (1)$$

$$a_{n+2i} = (2^{k-1} + i - 1)^2 + 2^k,\ \text{for } 1 \le i \le 2^{k-1} \quad (2)$$

where $n = n_k$. For $i = 2^{k-1}$, $n + 2i + 1 = n + 2^k + 1 = n_{k+1}$ and $a_{n_k+1}(2^{k-1} + 2^{k-1})^2 = (2^k)^2$. (1) and (2) show that a_m is not square for $n_k < m < n_{k+1}$.
