The Blakers Mathematics Contest 2003

Problems and Solutions

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1. An interesting sequence
For $n \geq 1$, define

$$H_n = \frac{1}{n} + \frac{1}{n+3} + \frac{1}{n+6} + \cdots + \frac{1}{n+3(n-1)}$$

Prove that $\{H_n\}$ converges and find its limit. [Hint: Consider Riemann sums.]

Solution (Shane Kelly, 2nd year, UWA)
Consider the function $f(x) = \dfrac{1}{1+3x}$
An upper Riemann sum for $f(x)$ over [0, 1] with interval size $1/n$ is

$$\begin{aligned}\sum_{i=1}^{n} f\left(\frac{1}{n}\right)\frac{1}{n} &= \sum_{i=1}^{n} \frac{1}{n+3i} = \sum_{i=1}^{n} \frac{1}{n+3(i-1)} - \frac{1}{n} \\ &= H_n - \frac{1}{n}\end{aligned}$$

Hence

$$lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{1}{n}\right)\frac{1}{n} = \int_0^1 \frac{1}{1+3x}\,dx = \frac{\ln 4}{3},$$

so $\lim_{n\to\infty} H_n = \dfrac{\ln 4}{3}$.

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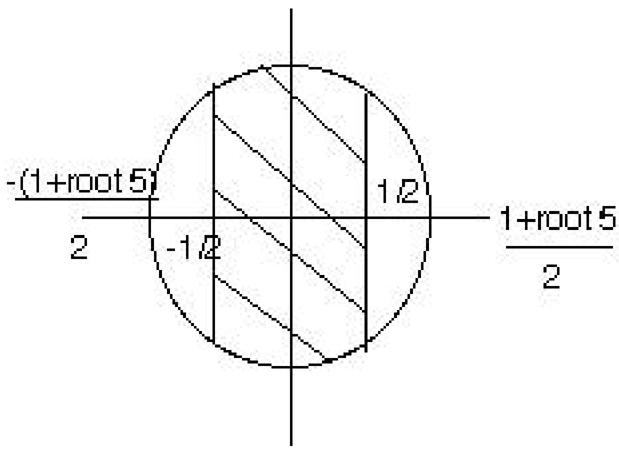
2. Range of a polynomial

What are all the possible values of solutions to $x^2 + px + q = 0$, where p and q can take on any values in the interval $[-1, 1]$?

Solution (Shreya Bhattarai, 2nd Year, UWA)
By the fundamental theorem of algebra, $x^2 + px + q = (x - z)(x - w)$ where $z,\ w \in \mathbb{C}$. Since the cofficients $p,\ q \in \mathbb{R}$, z and w have complex part conjugate, say $z = a + bi$ and $w = c - bi$.
Then $p = a + c \in [-1, 1]$ and $q = ac - b^2 \in [-1, 1]$. Furthermore, by continuity of polynomials and the Intermediate Value Theorem, all values satisfying these bounds can be attained.
Case 1 $b = 0$**:** We need the values of a for which $a + c \in [-1, 1]$ and $ac \in [-1, 1]$ for some c. Note that for $c = 0$, any $a \in [-1, 1]$ satisfies

these inequalities, and if (a, c) satisfies them, so does $(-a, -c)$. Hence for a maximum value of a, we only need consider $a > 1$.
If a is any solution, then $c \in [-a-1, -a+1]$ so $ac \in [-a(a+1), -a(a-1)]$. Hence the maximum value of a occurs at $-a^2 + a = -1$ so $a = \frac{1+\sqrt{5}}{2}$. By symmetry, the minimum is $\frac{-1-\sqrt{5}}{2}$ so the range is $|a| \leq \frac{1+\sqrt{5}}{2}$.
Case 2 $b \neq 0$ The solutions of the quadratic are complex conjugates, so $a = c$ and we have $p = 2a \in [-1, 1]$ and $a^2 + b^2 \in [-1, 1]$. The solution of this system is the region of the complex plane inside the unit disc between the verticals $Re(z) = -\frac{1}{2}$ and $Re(z) = \frac{1}{2}$, (not including the real axis).
Hence the range of the polynomial is shown in the diagram:



3. Dogs chasing each other

On a horizontal plane, n dogs $D_1, D_2, \ldots, D_n$ of equal mass are standing with their centres of mass at points $P_1, P_2, \ldots, P_n$. Initially, D_1 is facing D_2, D_2 is facing D_3,... D_n is facing D_1. Every dog turns through angle θ to its left. Then for some fixed positive real number k, D_1 moves forward a distance $k|P_1P_2|$; D_2 moves forward a distance $k|P_2P_3|; \ldots D_n$ moves forward a distance $k|P_nP_1|$, where $|P_1P_2|$ denotes the distance from P_1 to P_2.
Prove that after all the dogs have moved, their centre of mass is at the same point it was initially.

Solution (Michael Pauley, 2nd Year, UWA)

We are trying to prove that

$$\frac{1}{n}\sum_{i=1}^{n} k\,\mathrm{rot}_\theta(P_i - P_{i-1}) = 0,$$

that is, the displacement vector of the centre of mass is zero. Since initially the dogs form a closed circle we can assume

$$\sum_{i=1}^{n}(P_i - P_{i-1}) = 0,$$

where P_0 means P_n.
Since rotation and scaling fix the zero vector,

$$k\mathrm{rot}_\theta \sum_{i=1}^{n}(P_i - P_{i-1}) = \sum_{i=1}^{n} k\mathrm{rot}_\theta(P_i - P_{i-1}) = 0,$$

so $\frac{1}{n}\sum_{i=1}^{n} k\mathrm{rot}_\theta(P_i - P_{i-1}) = 0$, so the centre of mass does not change.

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4. Logical paradox

A well–known logical joke is a card saying on one side "The statement on the other side is false" and on the other, "The statement on the other side is true". Clearly, it is impossible for either of the statements to be true or false without leading to an inconsistency. However, if both sides said "The statement on the other side is false" or "The statement on the other side is true", it is possible for these statements to be logically consistent.

Now suppose you are given a stack of n cards, each of which says either "The statement on the next card is false" or "The statement on the next card is true". Is it possible to arrange the cards into an order which makes the whole pack logically consistent?

Solution

We assume that the last card in an arrangement refers to the first card. We are going to show that regardless of the order, the cards can be logically consistent if and only if the number of cards which state that the next card is false is even.

Suppose we have the deck of cards together with some consistent assignment of truth value to each card. Label each card by T if it says that the statement on the next card is true, and by F if it says that the statement on the next card is false. Also label each card by t if it is true, and by f if it is false. Thus each card has one of the labels Tt, Tf, Ft, or Ff.

We first show that any card labelled T can be removed from the deck without altering the remaining labels. If it is labelled Tt, the preceding card must be either Tt or Ff and the next card must be either Tt or Ft. Thus removing the given Tt card does not change anything in the rest. If it is labelled Tf, then the preceding card must be labelled Tf or Ft and next card must be either Ft or Ff. Once again, removing the given Tf card does not change the rest.

Hence we now assume that the remaining cards are all labelled Ft or Ff. Next we show that removing a pair of adjacent F cards does not alter the remaining labels. First note that a pair (Ft, Ft) cannot be consistent, since Ft truthfully claims that the statement on the next card is false. Similarly, (Ff, Ff) cannot be consistent, since Ff falsely claims that the statement on the next card is false, so that statement must be true. Hence any adjacent pair must be (Ft, Ff), in which case the preceding card is Ff and the next card is Ft; or (Ff, Ft) in which case the preceding card is Ft and the next card is Ff. In both cases, the pair can be removed.

Thus we finally reach a pair (Ft, Ff) or (Ff, Ft) which are consistent, or a single Ft or Ff which corresponds to 'The statement on the other side is false' which is inconsistent, a contradiction. Hence the original deck was consistent if there are an even number of F cards, and inconsistent if there are an odd number.

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5. Average distance

Let $[0,1]$ be be the unit interval. If $X = \{x_1,\, x_2, \ldots,\, x_n\}$ is an n–tuple of points in $[0,1]$ and t is any point in $[0,1]$, the **average distance** $D(t,\, X)$ from t to X is defined to be $\dfrac{1}{n}\displaystyle\sum_1^n |t - x_i|$.

Prove that for any choice of $n > 1$ and X in $[0,1]$, there exists $t \in [0,1]$ such that $D(t, X) = \dfrac{1}{2}$. Show also that $\dfrac{1}{2}$ is the only distance for which this result is true.

Now show that if $[0,1]$ is replaced by an equilateral triangle of side 1, and all distances are measured along the sides of the triangle, then the same results are true with $\dfrac{1}{2}$ replaced by $\dfrac{2+\sqrt{3}}{6}$.
(Apologies from the setters. The second part should have stated that the n–tuple of points lie on the sides of an equilateral triangle, but the distances measured are ordinary Euclidean distances in the plane)

Solution (Michael Pauley, 2nd Year, UWA)
Since all the numbers in X lie betrween 0 and 1, so does their average, A. If $A \le \frac{1}{2}$, then the average of $\{(1 - x_i)\} \ge \frac{1}{2}$ and vice versa. So either $D(0, X) \le \frac{1}{2}$ and $D(1, X) \ge \frac{1}{2}$ or vice versa. Since $D(t, X)$ is a continuous function of t, by the Intermediate Value Theorem there exists t such that $D(t, X) = \frac{1}{2}$.
To see that $\frac{1}{2}$ is the only possibility for all choices of X, let $X = \{0, 1\}$; in this case $D(t, X) = \frac{1}{2}$ for all $t \in [0, 1]$.
When we replace the interval by an equilateral triangle of side 1, the average distance function is still continuous. Taking X to be the three vertices and the three side midpoints, we see that $D(t, X) \le \dfrac{2+\sqrt{3}}{6}$ at some point and $D(t, X) \ge \dfrac{2+\sqrt{3}}{6}$ at some point, and hence by the IVT attains this value at some t.

To see that this is the only possibility, for any $d \neq \dfrac{2+\sqrt{3}}{6}$, place enough points t regularly around the triangle so that the average distance $D(t, X) \neq d$.

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6. Coin toss

Two fair coins are tossed simultaneously until at least one of the two shows a head. If only one of the two is a head, the other is tossed until it turns up head. What is the expected number of tosses to get heads on both the coins?

Solution (Shreya Bhattarai, 2nd Year, UWA)

Let $E(2)$ be the expected number of tosses to finish the game and $E(1)$ the expected number to toss one head.

Then $E(2) = 1 + \frac{1}{4}(E(2)) + \frac{1}{2}(E(1))$, where the first term represents the current toss, the second a toss of $\{T, T\}$, and the third a toss of $\{T, H\}$.

Since $E(1) = 1 + \frac{1}{2}(E(1))$ we see that $E(1) = 2$ and hence $E(2) = 2 + \frac{1}{4}(E(2))$, from which $E(2) = \frac{8}{3}$.

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7. Maximise the product

(a) Given a positive integer n, find a finite sequence $(a_1, a_2, \ldots, a_m)$ of positive integers whose sum is n and whose product is maximal.

(b) Now try the same problem with 'integer' replaced by 'real number'.

Solution (Shreya Bhattarai, 2nd year, UWA)

(a) Let $a_i + \cdots + a_k = n > 1$. If any $a_i \geq 4$, we do not decrease the product by replacing a_i by $(a_i - 2) + 2$, so we can assume all the a_i are 2 or 3. If there are more than two 2's, we can increase the product by replacing $2 + 2 + 2$ by $3 + 3$. Hence the maximum product is obtained with zero, one or two 2's and the rest of the terms 3's.

(b) First we show that for a maximum product, all the terms of the sum must be equal. Suppose $n = x_1 + \cdots x_k$ with $x_i \neq x_j$. By the arithmetic-geometric mean inequality, $\left(\dfrac{x_i + x_j}{2}\right)^2 > x_i x_j$ so the product can be increased by replacing each by their average.

Now we have to find the number k which maximises the product $\left(\dfrac{n}{k}\right)^k$.

For this, replace the rational n/k by a continuous variable x and consider the differentiable function $f(x) = x^{n/x}$.

Then

$$f'(x) = \left(\frac{n}{x^2} - \frac{n\ln x}{x^2}\right)x^{n/x} = \frac{n}{x^2}x^{n/x}(1-\ln x)$$

Since $f'(x) > 0$ for $x < e$ and $f'(x) < 0$ for $x > e$, $f(x)$ attains its maximum for $x = e$.
So n/k should be close to e for the maximum product. Choose the integer m such that $m < n/e < m+1$ and compare $(n/m)^m$ and $(n/m+1)^{m+1}$. Call m^* whichever of m or $m+1$ gives the greater value. Then the required sequence is constant with m^* terms of value n/m^*.

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8. 'Regression' line

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ be three points in the plane, with $x_1 < x_2 < x_3$. Say that $y = Ax + B$ is a *least absolute line* if the function

$$g(a, b) = \sum_1^3 |ax_i + b - y_i|$$

has a minimum at $(a, b) = (A, B)$.
Must a least absolute line pass through two of the three points?

Solution Yes. Each line $y = ax + b$ with $a \neq 0$ divides the plane into an upper half plane U_{ab} and a lower half plane L_{ab}.
For fixed a, $g(a, b)$ is a decreasing function of b when U_{ab} contains two or three P_i in its interior and an increasing function of b when L_{ab} contains two or three P_i in its interior.
So for fixed a, the minimum of $g(a, b)$ occurs when the line $y = ax + b$ contains a vertex (x_r, y_r) of triangle $P_1P_2P_3$ and a point on the opposite side.
For the line $y = a(x - x_r) + y_r$ containing (x_r, y_r), the function

$$g(a, y_r - ax_r) = \sum_{i=1}^3 |a(x_i - x_r) - (y_i - y_r)|$$

is a linear function of a if the line does not contain a second vertex. Also $\lim_{a\to\pm\infty} g(a, y_r - ax_r) = \infty$.
Hence if $y = ax+b$ contains (x_r, y_r) then the minimum of $g(a, b)$ occurs when the line contains a second vertex. A simple geometric argument shows that the distance from P_i to this line is minimised when $i = 2$. Thus the least absolute line is that containing P_1 and P_3.

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9. Circulant matrices

A matrix such as

$$\begin{bmatrix} 2 & 5 & 8 & 9 \\ 9 & 2 & 5 & 8 \\ 8 & 9 & 2 & 5 \\ 5 & 8 & 9 & 2 \end{bmatrix}$$

in which each row is obtained by shifting the previous row one step to the right, with the last entry returning to the beginning, is called a **circulant**.

Prove that the set of $n \times n$ circulant matrices with real number entries is closed under addition and multiplication and multiplication is commutative.

Show that if a circulant is invertible, then its inverse is a circulant.

Solution Let $A = (a_{ij})$ and $B = (b_{ij})$ be circulant matrices, so $a_{ij} = a_{i+1,j+1}$ (indices taken modulo n). Then $a_{ij} + b_{ij} = a_{i+1,j+1} + b_{i+1,j+1}$ so $A + B$ is circulant.

Now $AB = (c_{ij})$ where $c_{ij} = \sum_k a_{ik}b_{kj} = \sum_k a_{i+1,k+1}b_{k+1,j+1} = c_{i+1,j+1}$. Hence AB is circulant.

Also $BA = (d_{ij})$ where

$$\begin{aligned} d_{ij} &= \sum_k b_{ik}a_{kj} = \sum_k b_{i+(j-k),k+(j-k)}a_{k+(i-k),j+(i-k)} \\ &= \sum_k b_{i+j-k,j}a_{i,i+j-k} = \sum_k a_{i,i+j-k}b_{i+j-k,j} = c_{ij}. \end{aligned}$$

Hence $BA = AB$.

Now suppose A is an invertible circulant with first row $(a_1, a_2, \ldots, a_n)$. We have to show that there is an n–tuple $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ such that if B is the circulant with this first row, then $AB = 1$. The first column of B is also $\mathbf{b}$, so $\mathbf{b}$, if it exists, is the solution of the linear system with augmented matrix $[A|e_1]$, where e_1 is the first standard basis element. Since A is invertible, $\mathbf{b}$ exists (and is unique). But then the second column of B, $(b_2, \ldots, b_n, b_1)$ is the solution of the linear system with augmented matrix $[A|e_2]$ and so on. Hence $AB = I_n$.

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10. Jumping Gorillas

A gorilla, of mass 100kg, is imprisoned inside a cage of mass 350kg, which is suspended in mid-air on a long rope thrown over a pulley. At the other end of the rope is a counterweight of mass 450kg, exactly balancing the combined weight of the cage and gorilla.

Attached to the roof of the cage, on the inside, is a delicious bunch of bananas, which the gorilla would love to eat. Unfortunately they're

out of his reach, by 10cm. He decides to jump up into the air, just far enough to be able to grab the bananas, then fall back down to the cage floor.
How fast does he need to jump? Specifically, with what speed does the gorilla need to move away from the cage floor?
You may assume that $g = 10ms^{-2}$.
(We acknowledge with thanks the on–line magazine Plus (http://plus.maths.org.uk/) from which this problem was borrowed.)

Solution Assume that the rope is light, the pulley frictionless etc. For convenience, we can regard the system (cage + counterwieght) as a single element.
In jumping, the gorilla exerts a force internal to the system (gorilla + cage + counterwieght) so in this phase of the motion the centre of mass does not move.
Let the initial velocity of the gorilla upwards be u and the initial velocity of the (cage + counterwieght) downwards be U.
The fixed centre of mass implies $100u = (350 + 450)U$, so $u = 8U$.
Choose the origin at the floor of the cage and measure displacements upwards.
For the subsequent motion of the gorilla we have

$$\begin{aligned} \ddot{x} &= -g \\ \dot{x} &= u - gt, \ (\dot{x}(0) = u) \\ x &= ut - \frac{1}{2}gt^2, \ (x(0) = 0) \end{aligned}$$

The subsequent motionof the cage has upwards acceleration

$$\frac{\text{difference of masses}}{\text{sum of masses}} = \frac{100}{8}g$$

Then

$$\begin{aligned} \ddot{X} &= g/8 \\ \dot{X} &= -U + gt/8, \ (\dot{X}(0) = -U) \\ X &= -Ut + \frac{1}{2}gt^2/16, \ (X(0) = 0) \end{aligned}$$

Distance between gorilla and cage

$$
\begin{aligned}
&= x - X \\
&= ut - \frac{1}{2}gt^2 - (-Ut + gt^2/16) \\
&= \frac{9}{8}(ut - \frac{1}{2}gt^2
\end{aligned}
$$

This attains its maximum when $t = \dfrac{u}{g}$ so the maximum distance is $\dfrac{9}{8}\dfrac{1}{2}\dfrac{u^2}{g}$.

We require this distance to be 0.1 m, implying $u = \sqrt{0.1\dfrac{16g}{9}} = \dfrac{4}{3}$, using $g = 10ms^{-2}$.

Then the relative velocity of gorilla and cage is $u + U = \frac{9}{8}u = 1.5ms^{-1}$.