The Blakers Mathematics Contest 2007

SOLUTIONS

1. Mean and Median

Consider n distinct real numbers $x_1, \ldots, x_n$. What is the minimal number n of numbers such that the mean $\bar{x}$ equals the median $\tilde{x}$ but the points are not symmetric about the mean?

Solution:
Without loss of generality, assume $x_1 < x_2 < \ldots < x_n$. For $n = 1$ and $n = 2$ the mean and median coincide and all sets of size 1 and 2 are symmetric about the mean.

For $n = 3$ the median equals x_2, hence the mean equals x_2 if and only if $x_2 - x_1 = x_3 - x_2$, i.e., the points are symmetric about the mean.

For $n = 4$ assume by shifting and scaling that $x_2 = -1$ and $x_3 = 1$, so the median equals 0. Then the mean equals zero if and only if $x_1 = -x_4$, so once again the points are symmetric.

For $n = 5$ the set $\{-4, -3, 0, 2, 5\}$ serves as a counterexample, so $n = 5$.

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2. Even and odd functions

A real–valued function f on an interval $[a, b]$ is called even if $f(a + x) = f(b - x)$ for all $x \in [0,\, b - a]$ and odd if $f(a + x) = -f(b - x)$ for all $x \in [0,\, b - a]$.

Show that every real function on $[a, b]$ is the sum of an even function and an odd function. Is such a sum unique?

Solution: Let $f(x)$ be a real–valued function on $[a, b]$. Then $f(x) = \dfrac{f(x) + f(a + b - x)}{2} + \dfrac{f(x) - f(a + b - x)}{2}$, the first term being even and the second odd.

This representation is unique because if $f = g + h$ with g even and h odd, then for all $x \in [a, b]$,

$$\begin{aligned}
f(x) &= g(x) + h(x)\\
f(a + b - x) &= g(a + b - x) + h(a + b - x)\\
&= g(x) - h(x)
\end{aligned}$$

Hence $g(x) = \dfrac{f(x) + f(a+b-x)}{2}$ and $h(x) = \dfrac{f(x) - f(a+b-x)}{2}$.

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3 Five Villages in England

The following linear distances of five villages in England are:

- 30km from Ayling to Beeling
- 80km from Beeling to Ceiling
- 236km from Ceiling to Dealing
- 86km from Dealing to Ealing
- 40km from Ealing to Ayling.

What is the linear distance from Ealing to Ceiling? All distances are measured on a map.

Solution: These five villages form a closed polygon with the special property that one edge is the sum of the other four edges: Ceiling to Dealing $= 236 = 30+80+86+40$. Hence, all corners of the polygon are on a line. The order of the cities is: Dealing, Ealing, Ayling, Beeling and Ceiling. Hence, the distance from Ealing to Ceiling is $40 + 30 + 80 = 150$km.

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4. Sphere in a Tetrahedron

Let A, B, C and D be four non-planar points in 3–space.

(a) Prove that there is a unique sphere tangent to each of the triangles ABC, BCD, ACD and ABD.

(b) Call this sphere the insphere of the tetrahedron $ABCD$, and let its radius be r. Let h_A, h_B, h_C and h_D denote the altitudes of the tetrahedron, i.e., distances from each vertex to the plane determined by the triangle opposite that vertex. Prove that

$$\frac{1}{r} = \frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C} + \frac{1}{h_D}$$

Solution: (a) Consider $ABCD$ as a hollow tetrahedron. Remove one triangular face and drop in a sphere small enough so that it sits in the opposite corner touching all three faces but does not project outside the removed face. Now expand the sphere until it just touches the fourth face. It is then tangent to all faces of the tetrahedron. This sphere is unique, for if there is another with the same property, it cannot have the same centre but a different radius, otherwise it would fail to meet or project beyond the faces of the tetrahedron; if it had a different centre, it would be closer to at least one face and farther from at least one face and hence could not be tangent to all faces.

(b) Let ΔXYZ denote the area of any triangle XYZ. The volume V of the tetrahedron is the one third the area of any face times the distance to the opposite vertex. Hence

$$V = \frac{1}{3}h_A \Delta BCD = \frac{1}{3}h_B \Delta ACD = \frac{1}{3}h_C \Delta ABD = \frac{1}{3}h_D \Delta ABC.$$

But V is also the sum of the volumes of the four tetrahedra whose vertices are three of the given points and the centre of the insphere, i.e., $V = \frac{1}{3}r(\Delta BCD + \Delta ACD + \Delta ABD + \Delta ABC)$.

Hence $\dfrac{3V}{r} = \Delta BCD + \Delta ACD + \Delta ABD + \Delta ABC =$
$3V\left(\dfrac{1}{h_A} + \dfrac{1}{h_B} + \dfrac{1}{h_C} + \dfrac{1}{h_D}\right)$.

Hence

$$\frac{1}{r} = \frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C} + \frac{1}{h_D}$$

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5. A stable table

Legs $L_1,\ L_2,\ L_3$ and L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4–tuples $(k_1,\ k_2,\ k_3,\ k_4)$ of non–negative integers can we cut a piece of length k_i from the end of leg $L_i,\ i = 1,\ 2,\ 3,\ 4$ and still have a stable table? (i.e. one that can be placed so that all four leg ends touch the floor.)

Solution: Turn the table upside down so that its surface lies in the $x - y$ plane with corners with legs $L_1,\ L_2,\ L_3,\ L_4$ at $(1, 0),\ (0,\ 1),\ (-1,\ 0)$ and $(0,\ -1)$ respectively. Let ℓ_i be the length of L_i after the cut, so that $0 \le \ell_i \le n$ for $i = 1,\ 2,\ 3,\ 4$.

The table will be stable if and only if the four ends are coplanar. This will happen if and only if the diagonals joining the leg ends intersect at the mid–point of each. That is, if and only if

$$(*) \qquad \left(0, 0, \frac{\ell_1 + \ell_3}{2}\right) = \left(0, 0, \frac{\ell_2 + \ell_4}{2}\right).$$

Thus the third coordinate in each of these expressions can be any of $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, n$.

For each $0 \le k \le n$, let S_k be the number of integer solutions of $x + y = k,\ 0 \le x,\ y \le n$. Then the number of solutions of $(*)$, i.e. the number of stable tables, is $N = \sum_{k=0}^{n} S_k^2$.

Thus

$$
\begin{aligned}
N =& 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 + n^2 + \cdots + 1^2 \\
=& 2\frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
=& \frac{1}{3}(n+1)(2n^2 + 4n + 3).
\end{aligned}
$$

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6. An integer quadrilateral

ABCD is a quadrilateral and P is an interior point. Lines are drawn from each vertex to P. The side lengths of the four non–congruent triangles are integers not greater than 20. The angles at P are 60°, 120°, 90°, 90° in that order. What is the perimeter of the quadrilateral ABCD?

Solution: The right angled triangles must have sides chosen from the triads (5, 12, 13), (5, 8, 17) and multiples of (3, 4, 5). Since they must have one side (not a hypotenuse) in common, the possibilities are

(i) (5, 12, 13) and (12, 16, 20)
(ii) (6, 8, 10) and(8, 15, 17)
(iii) (9, 12, 15) and (12, 16, 20)
(iv) (5, 12, 13) and (9, 12, 15)

Case (i): the quadrilateral must be as shown in Figures 1 or 2.

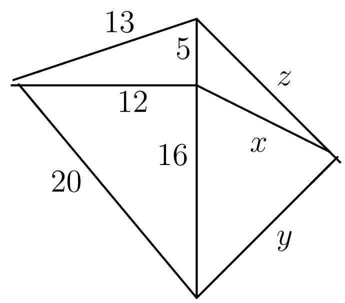


Figure 1

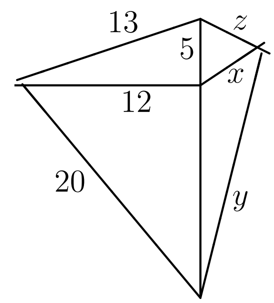


Figure 2

In Figure 1

(1) $$y^2 = 16^2 + x^2 - 32x\cos 60 = 256 + x^2 - 16x$$

(2) $$z^2 = 5^2 + x^2 - 10x\cos 120 = 25 + x^2 + 5x$$

From (1), $x = 8 \pm \sqrt{y^2 - 192}$

For x real, $y^2 > 192$ implies $y \geq 14$. Testing $y = 14, 15, \ldots, 20$ shows the only values giving integer values for x are $y = 14,\ x = 6$ or 10, $y = 16,\ x = 0$ or 16, and $y = 19,\ x = -5$ or 21.

Of the acceptable values for x, only $x = 16$ yields an integer $z = 19$.

Hence the perimeter is $13 + 20 + 16 + 19 = 68$.

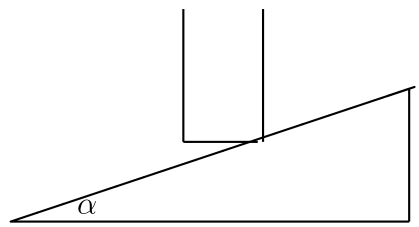
A similar analysis for Figure 2 yields $x = 5,\ y = 19,\ z = 5$ and the perimeter is 57.

Analysis of cases (ii), (iii) and (iv) leads to degenerate triangles (side lengths ≤ 0) or non–integral side lengths.

Hence the perimeter of the quadrilateral is 57 or 68.

7. A door-stop wedge

Obtain a sufficient condition on the angle of a door-stop wedge, in terms of friction characteristics between the wedge, the floor and the door, for the wedge to hold the door in place.



Consider the contact forces between the door and the wedge and between the wedge and the floor. It should be noted that the force exerted by the door on the wedge is usually quite large compared with the weight of the wedge.

Solution:

Figure 1 shows the forces acting on the wedge. The action of the door on the wedge tends to move the wedge to the right, opposed by the friction force due to the floor.

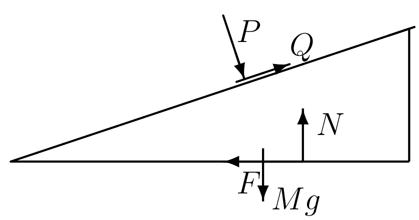


Figure 1

Resolving horizontally and vertically we obtain the equations of static equilibrium:

(i) $$P\cos\alpha - Q\sin\alpha = N - Mg$$

(ii) $$P\sin\alpha + Q\cos\alpha = F$$

Let $\mu_F = \tan\phi$ = coefficient of friction between the wedge and the floor.

Let $\mu_D = \tan\theta$ = coefficient of friction between the wedge and the door.

Then $F < N\tan\phi$ and $Q < P\tan\theta$ are conditions for equilibrium. The regions in the NF-plane and PQ-plane given by these inequalities are shown in Figure 2(a), (b).

We need to relate together these two regions which indicate where equilibrium is possible.

Solving equations (i) and (ii) for P and Q or resolving in the directions of P and Q gives

$$P = (N - Mg)\cos\alpha + F\sin\alpha \tag{iii}$$

$$Q = -(N - Mg)\sin\alpha + F\cos\alpha \tag{iv}$$

These equations show that with an origin shift, the PQ-axes are rotated through α in the NF-plane.

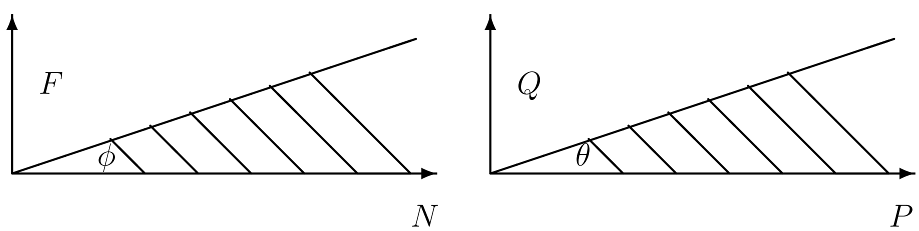


Figure 2(a) Figure 2(b)

(For example check that when $N = Mg,\ F > 0$ then $P > 0,\ Q > 0$)

Figure 3(a), (b) shows the orientation of the axes together with the regions which are the intersections of the angles of Figure 2(a), (b).

In Figure 3(a) a large value of P might not lie in the shaded region. This means that equilibrium under a large value of P is not possible i.e. the wedge would move.

In Figure 3(b) we see that any value of P is possible in an equilibrium situation i.e. the door-stop would be effective.

Hence a sufficient for an effective door-stop wedge is $\alpha < \phi$.

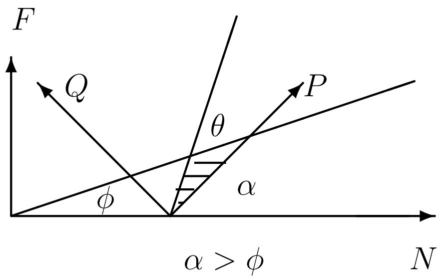


Figure 3(a)

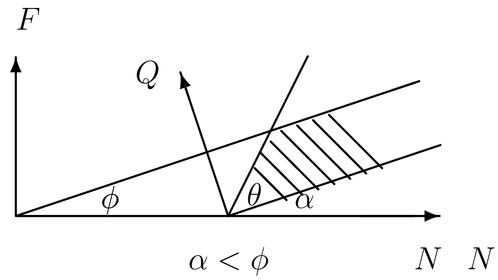


Figure 3(b)

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8. Bug in a Wineglass

A wineglass has internal shape obtained by rotating the function $y = x^2$ about the y–axis, for $x = 0$ to $x = D$. You pour the wine in at a constant rate of $\lambda\ l/sec$. On the inside of the glass sits a bug who is fairly unhappy about getting her feet wet, so begins crawling upward along the curve $y = x^2$ as soon as the liquid reaches her. How fast does she have to crawl to outrun the rising tide of liquid?

Solution: The volume of wine in the glass as a function of time t is $V(t) = \lambda t$. The volume when the liquid is at height h is $\phi(h) = \int_0^h \pi y dy = \pi h^2/2$.

Therefore the height of liquid at time t is $h(t) = \sqrt{2\lambda t/\pi}$.

The bug is crawling along the curve $x = \sqrt{y}$. Assuming she starts from height $h_0 > 0$, the distance she covers in time t is

$$L(t) = \int_{h_0}^{h(t)} \sqrt{1 + ((\sqrt{y})')^2} dy = \int_{h_0}^{h(t)} \sqrt{1 + \frac{1}{4y}} dy.$$

Her speed is $dL/dt = (1 + 1/4h)^{1/2}(dh/dt)$.

From $\lambda t = \pi h^2/2$ we obtain $dh/dt = \lambda/(\pi h)$. hence the minimum speed required to keep pace with the rising liquid is

$$\frac{dL}{dt} = \left(1 + \frac{1}{4h}\right)^{1/2} \frac{\lambda}{\pi h} = \left(1 + \frac{1}{4}\sqrt{\frac{\pi}{2\lambda t}}\right)^{1/2} \left(\frac{\lambda}{2\pi t}\right)^{1/2}$$

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9. A game of patience

A standard deck of 52 cards is arranged at random face up in 4 rows and 13 columns. Show that by finitely many swaps of two cards of the same value (e.g., 7♣ and 7♡) the cards can be re–arranged so that each column contains one club, one heart, one spade and one diamond.

Solution: Denote the cards by (x, i), where x is the suit and i the value. Suppose that some suit y is missing from the first column. Then some suit x occurs more than once in the first column. As long as x occurs more than once, switch (x, i) with (y, i), which must occur in another column. Do this until the 4 suits occur in column 1.

Now suppose that all suits occur in the first $k-1$ columns, but some suit y is missing from column k. Some suit x occurs more than once in column k. In at most k moves, we swap some (x, i) in column k for some (y, j) without eliminating any suits from the first $k-1$ columns.

Define a directed graph with 13 edges whose vertices are the columns as follows. For each value i draw an arrow from the column containing (x, i) to the column containing (y, i). Each of the first k columns (vertices) has one head and one tail and column k has at least two tails and no head, Hence there is a path from column k which may go through some of the first k columns, but eventually reaches a column later than k. Swap all the cards at heads and tails of arrows on this path, starting at column k. This will introduce suit y to column k but will not change the suits in columns up to $k-1$. Doing this at most 3 times introduces all suits to column k.

By induction, this works for columns 1 to 12, and the last column comes for free.

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10. Square in a cube

What is the area of the largest square contained in the unit cube?

Solution: For convenience in calculation, first consider a cube C centred at the origin with vertices at $(\pm1, \pm1, \pm1)$. The largest square contained in C has orthogonal diagonals centred at the origin with endpoints on the faces of C, so it suffices to find the longest such line segments.

Let A be a line segment with endpoints $(1/2, 1, 1)$ and $(-1/2, -1, -1)$ and B a line segment with endpoints $(1, 1/2, -1)$ and $(-1, -1/2, 1)$. A and B are orthogonal and have length 3.

Suppose there are orthogonal line segments X and Y in C passing through the origin of length > 3. By symmetry, we can assume one endpoint of X is $(1, y, z)$ with $0 \leq y \leq z \leq 1$. Since the length of $X > 3$, it follows that $y > 1/2$. Three possible endpoints of Y are $P = (-1, u, v),\ (u, -1, v)$ or $(u, v, -1)$ with $|u|,\ |v| \leq 1$.

Suppose $P = (-1.u.v)$. Since Y is orthogonal to X, $yu+zv = 1$. In the (u, v)-plane we are considering points on the line Y within the square $|u|,\ |v| \leq 1$ and we want to maximise u^2+v^2. This maximum occurs on the boundary of the square, so at the points $u = 1,\ v = (1-y)/z$ and $u = (1-z)/y,\ v = 1$. Since $0 \leq y \leq z \leq 1,\ 1 \geq (1-y)/z \geq (1-z)/y \geq 0$, hence we have

$$|Y|^2 = 4((1+u^2+v^2) \leq 4\left(2+\left(\frac{1-y}{z}\right)^2\right)$$

If $|Y| > 3$, this yields $(1-y)/z > 1/2$ and hence $y < 1 - z/2$.

Since $|X| > 3,\ y^2+z^2 > 5/4$ and hence

$$\left(1-\frac{z}{2}\right)^2 + z^2 > y^2 + z^2 > \frac{5}{4}$$

There is no solution to this inequality with $0 \leq z \leq 1$. Hence 3 is the maximum length of $|X|$ and $|Y|$

A similar calculation works for the other two possible values of P.

Now return to the original problem by halving all lengths. The diagonal of the largest square in the cube is $3/2$ so its area is $9/8$.