The Blakers Mathematics Contest 2009 Solutions

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1. Interesting function

Let $y = x^{\frac{1}{x}}$, $x > 0$. Find the intervals on which $y(x)$ is monotonic, and on each such interval, find its range.

Solution: [Robert Palmer, Y2, UWA] Since $x > 0$ implies $y > 0$, we can take logs on both sides to conclude

$$\ln y = \frac{1}{x} \ln x$$

Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2}(1 - \ln x)$$

Since y and $x^2 > 0$, the only crtical point is at $\ln x = 1,\ x = e$.

Furthermore, $\frac{dy}{dx}$ is positive for $0 < x < e$ and negative for $e < x < \infty$.

Hence $y(x)$ is increasing on $(0, e]$ with range $(0, e^{1/e}]$ and decreasing on $[e, \infty)$ with range $[e^{1/e}, 1)$.

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2. Approximate area

A segment of a circle is the smaller region cut off the circle by a chord. The apothem of the segment is the line segment perpendicular to the chord from the midpoint of the chord to the perimeter of the segment. Two methods for approximating the area of the segment have been proposed:

(a) The trapezoidal approximation is the area of the trapezoid whose base is the chord, and upper edge the tangent to the segment at its midpoint with length the length of the apothem.

(b) The parabolic approximation is the area of the segment of the parabola passing through the endpoints of the chord and the vertex of the segment.

Show that the trapezoidal approximation is more accurate than the parabolic when the segment is 'large' (approaches a semicircle), while the parabolic approximation is more accurate when the segment is 'small'.

Find the segment for which the trapezoidal and parabolic approximations are equal.

Solution: Without loss of generality, assume the radius is 1. Let c be the length of the chord and h the length of the apothem and θ half the angle subtended by the chord at the centre of the circle.

Then the trapezoidal approximation is $A(t) = \frac{1}{2}(c+h)h$ and the parabolic approximation is $A(p) = \frac{2}{3}ch$.

These are equal when $h = \frac{1}{3}c$, and in this case, $A(t) = A(p) = \frac{2}{9}c^2$.

The actual area $A = \theta - \sin\theta\cos\theta$.

As $\theta \to \frac{\pi}{2}$, $c \to 2$ and $h \to 1$ so $A(t) \to \frac{3}{2}$, $A(p) \to \frac{4}{3}$ while $A \to \frac{\pi}{2} \approx 1.57$, so the trapezoidal approximation wins for large segments.

For the limit as $\theta \to 0^+$ we need to express $A(t)$ and $A(p)$ in terms of θ. Since $c = 2\sin\theta$ and $h = 1 - \cos\theta$,

$$\begin{aligned}
A(t) &= \frac{1}{2}(2\sin\theta + 1 - \cos\theta)(1-\cos\theta) \\
&= \frac{1}{2}(2(\sin\theta - \cos\theta) - \sin 2\theta + \cos^2\theta + 1) \text{ and} \\
A(p) &= \frac{2}{3}(2\sin\theta)(1-\cos\theta) = \frac{2}{3}(2\sin\theta - \sin 2\theta).
\end{aligned}$$

Since each of $\lim_{\theta\to 0^+} A$, $\lim_{\theta\to 0^+} A(t)$ and $\lim_{\theta\to 0^+} A(p) = 0$, we must use L'Hospital's Rule to compute $\lim_{\theta\to 0^+} A/A(t)$ and $\lim_{\theta\to 0^+} A/A(p)$.

To simplify the notation, we first compute derivatives of each of the three functions until the limit is non–zero.

$$\begin{aligned}
A &= \theta - \frac{1}{2}\sin 2\theta \to 0,\ A' = 1 - \cos 2\theta \to 0,\ A'' = 2\sin 2\theta \to 0, \\
A''' &= 4\cos 2\theta \to 4. \\
A(t) &= \frac{1}{2}(2\sin\theta - \sin 2\theta - 2\cos\theta + \cos^2\theta + 1) \to 0, \\
A(t)' &= \cos\theta + \sin\theta - \cos 2\theta - \frac{1}{2}\sin 2\theta \to 0, \\
A(t)'' &= -\sin\theta + \cos\theta + 2\sin 2\theta - \cos 2\theta \to 0, \\
A(t)''' &= -\cos\theta - \sin\theta + 4\cos 2\theta + 2\sin 2\theta \to 3. \\
A(p) &= \frac{2}{3}(2\sin\theta - \sin 2\theta) \to 0,\ A(p)' = \frac{2}{3}(2\cos\theta - 2\cos 2\theta) \to 0, \\
A(p)'' &= \frac{2}{3}(-2\sin\theta + 4\sin 2\theta) \to 0,\ A(p)''' = \frac{2}{3}(-2\cos\theta + 8\cos 2\theta) \to 4.
\end{aligned}$$

So miraculously, all three third derivatives have non–zero limits so by L'Hospital's Theorem, $\lim_{\theta\to 0^+} A/A(t) = 4/3$ while $\lim_{\theta\to 0^+} A/A(p) = 1$.

Hence the parabolic approximation wins for small segments.

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3. Matrix Inverse

Let A and B be $n \times n$ real matrices satisfying
(a) $AB = BA$
(b) $A^2 = A$ and $B^2 = B$
(c) $A - B$ is invertible.
Prove that $A + B = I_n$, the identity matrix.

Solution: [Aaran Mohann, Y2, Curtin]

$$\begin{aligned}(A+B)(A-B) &= A^2 - AB + BA - B^2\\ &= A^2 - B^2 \text{ (since } AB = BA)\\ &= A - B \text{ (since } A = A^2 \text{ and } B = B^2).\end{aligned}$$

Since $A - B$ is invertible,

$$\begin{aligned}A + B &= (A+B)(A-B)(A-B)^{-1}\\ &= (A-B)(A-B)^{-1} = I_n.\end{aligned}$$

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4. Integer divisors

(a) Prove that if x and y are positive integers, then xy divides $x^2 + y^2$ if and only if $x = y$.
(b) Prove that there are infinitely many triples (x, y, z) of positive integers such that $xyz = x^2 + y^2 + z^2$.

Solution: (a) xy divides $x^2 + y^2$ if and only if there is a positive integer a such that $x^2 + y^2 = axy$.

Consider this as a quadratic in x. It has an integer solution if and only if the discriminant $a^2y^2 - 4y^2$ is a perfect square, which is true if and only if $a^2 - 4$ is a perfect square. Since a is positive, this holds if and only if $a = 2$, which is the case if and only if $x = y$.

(b) $(x, y, z) = (3, 3, 3)$ is clearly a solution, and if (a, b, c) is any solution with $a \le b \le c$, then so is $(b, c, bc - a)$ because $b^2 + c^2 + (bc - a)^2 = b^2c^2 - abc = bc(bc - a)$. Since $bc - a >$

c, this leads to an infinite sequence of solutions, for example $(3,3,3),\ (3,3,6),\ (3,6,15)\ldots.$

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5. The camel inheritance

An old man willed that upon his death, his three sons would receive the uth, vth and wth parts of his herd of camels (i.e., $1/u, 1/v$ and $1/w$ of the herd). He had $uvw-1$ camels in his herd when he died. Since they could not divide $uvw-1$ into $u,\ v$ and w parts, they approached a distinguished mathematician for help. He rode over and added his camel to the herd, and then fulfilled the old man's wishes. One camel remained, which was of course his own.

How many camels were there in the herd?

Solution: The answer is 41. We may assume that $u \le v \le w$, and of course $u \ge 2$, otherwise one son would get all the camels. Then the condition of the will is that $vw + uw + uv = uvw - 1$.

(a) If $u = 2$ then the equation becomes $(v-2)(w-2) = 5$ which has the unique integral solution $v = 3,\ w = 7$ and $uvw - 1 = 41$.

(b) If $u = 3$, the equation becomes $(2v-3)(2w-3) = 11$, which has unique solution $v = 2,\ w = 7$. But his violates the condition $u \le v$.

(c) If $u \ge 4$ then we have $1 - 1/(uvw) = 1/u + 1/v + 1/w \le 3/4$, so $uvw \le 4$ which is impossible.

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6. A functional equation

let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying

$$f(x,y) + f(y,z) + f(z,x) = 0 \text{ for all real numbers } x,\ y \text{ and } z$$

Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) - g(y)$ for all real numbers x and y.

Solution: [Robert Palmer, Y2, UWA] From $(x,y,z) = (0,0,0)$, we obtain $3f(0,0) = 0$ and hence $f(0,0) = 0$.

From $(x,y,z) = (0,a,0)$, we obtain for all $a \in \mathbb{R},\ f(0,a) + f(0,0) + f(a,0) = f(a,0) + f(0,a) = 0$, or $f(0,a) = -f(a,0)$.

Hence for all $x,\ y \in \mathbb{R},\ f(x,y) + f(y,0) - f(x,0) = 0.$

Define $g(x) = f(x,0)$. Then for all $x,\, y \in \mathbb{R}$, $f(x,y) = g(x) - g(y)$.

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7. Average speed

A racing motorist completed n laps of a course at average lap speeds of $v_1 \leq v_2 \leq \cdots \leq v_n$. Show that the average speed for the n laps satisfies $v_1 \leq V < nv_1$, but there is no lower bound on V as a function of v_n.

Solution: Clearly $v_1 \leq V$. Suppose the length of the course is d. Then

$$V = \frac{\text{total distance}}{\text{total time}} = \frac{nd}{d\sum_{j=1}^{n}\frac{1}{v_j}} = \frac{nv_1}{\sum_{j=1}^{n}\frac{v_1}{v_j}}$$

But $\sum_{j=1}^{n} \dfrac{v_1}{v_j} > 1$ and $V = \dfrac{nv_1}{\sum_{j=1}^{n}\frac{v_1}{v_j}} < nv_1$.

There is no such lower bound, because for any speeds on the first $n-1$ laps, the last lap can be made sufficiently slowly to make V as small as you like.

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8. An Improper Integral

Show that

$$\lim_{T\to\infty}\int_1^T \sin(x)\sin(x^2)\,dx$$

converges.

Solution:

Integrating by parts

$$\int_1^T \sin(x)\sin(x^2)\,dx$$

$$= \left(\frac{\sin x}{x}\right)\left(\frac{-\cos x^2}{2}\right) - \int \left(\frac{-\cos x^2}{2}\right)\left(\frac{-\sin x}{x^2} + \frac{\cos x}{x}\right)dx$$

$$= \left(\frac{\sin x}{x}\right)\left(\frac{-\cos x^2}{2}\right) - \int \frac{\cos x^2 \sin x}{2x^2}\,dx$$

$$+ \frac{\cos x}{2x^2}\left(\frac{\sin x^2}{2}\right) - \int \left(\frac{\sin x^2}{2}\right)\left(\frac{1}{2}\left(\frac{-\sin x}{x^2} - \frac{2\cos x}{x^3}\right)\right)dx$$

Since

$$\lim_{T\to\infty}\frac{\sin T\cos T^2}{T}=0=\lim_{T\to\infty}\frac{\cos T\sin T^2}{T}$$

we only have to show that

$$\int_1^T\frac{\cos x^2\sin x}{x^2}\,dx,\quad \int_1^T\frac{\sin x^2\sin x}{x^2}\,dx,\quad\text{and}\quad \int_1^T\frac{\sin x^2\cos x}{x^3}\,dx$$

all converge as $T\to\infty$. But all three converge absolutely since $\int_1^T\frac{1}{x^2}\,dx$ and $\int_1^T\frac{1}{x^3}\,dx$ both converge as $T\to\infty$.

Alternative Solution: [Wilson Ong, Y3, UWA]

By Euler's identity, $e^{ix^2}=\cos x^2+i\sin x^2$.

$\int_0^\infty e^{ix^2}\,dx=\int_0^\infty\cos x^2\,dx+i\int_0^\infty\sin x^2\,dx$ converges since the Fresnel integrals $\int_0^\infty\cos x^2\,dx$ and $\int_0^\infty\sin x^2\,dx$ both converge.

$e^{ix}=\cos x+i\sin x$ and $e^{-ix}=\cos x-i\sin x$ imply $\sin x=\frac{e^{-ix}-e^{ix}}{2}i$.

Now

$$\begin{aligned}
\int_1^\infty\sin x\sin x^2\,dx &= \Im\left[\int_1^\infty\sin x(i\sin x^2)\,dx\right]\\
&= \Im\left[\int_1^\infty\sin x(e^{ix^2}-\cos x^2)\,dx\right]\\
&= \Im\left[\int_1^\infty e^{ix^2}\sin x\,dx\right]\\
&= \Im\left[\int_1^\infty e^{ix^2}(\tfrac{e^{-ix}-e^{ix}}{2}i)\,dx\right]\\
&= \frac{1}{2}\Im\left[i\int_1^\infty(e^{ix^2-ix}-e^{ix^2+ix})\,dx\right]\\
&= \frac{1}{2}\Im\left[i(e^{-i/4}\int_1^\infty e^{i(x-1/2)^2}\,dx-e^{-i/4}\int_1^\infty e^{i(x+1/2)^2}\,dx)\right]
\end{aligned}$$

Observe that $\int_1^\infty e^{i(x-1/2)^2}\,dx=\int_{1/2}^\infty e^{iu^2}\,du=\int_0^\infty e^{iu^2}\,du-\int_0^{1/2}e^{iu^2}\,du$ is convergent since $\int_0^\infty e^{iu^2}\,du$ and $\int_0^{1/2}e^{iu^2}\,du=\int_0^{1/2}\cos u^2\,du+i\int_0^{1/2}\sin u^2\,du$ are convergent.

Similarly, $\int_1^\infty e^{i(x+1/2)^2}\,dx=\int_{3/2}^\infty e^{iu^2}\,du=\int_0^\infty e^{iu^2}\,du-\int_0^{3/2}e^{iu^2}\,du$ is convergent.

Hence $\int_1^\infty \sin x \sin x^2 \, dx$ converges.

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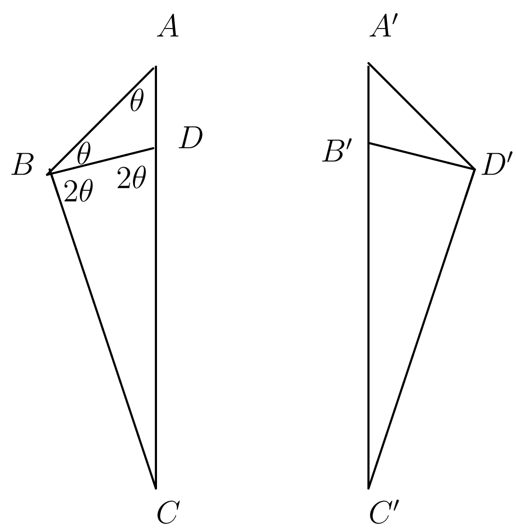
9. Cakes and Boxes

A triangular cake and a triangular box are congruent, but mirror images of each other. We would like to cut the cake into two pieces which can fit together in the box without turning either piece over.

(a) Show that this is possible if one angle of the triangle is three times as large as another.

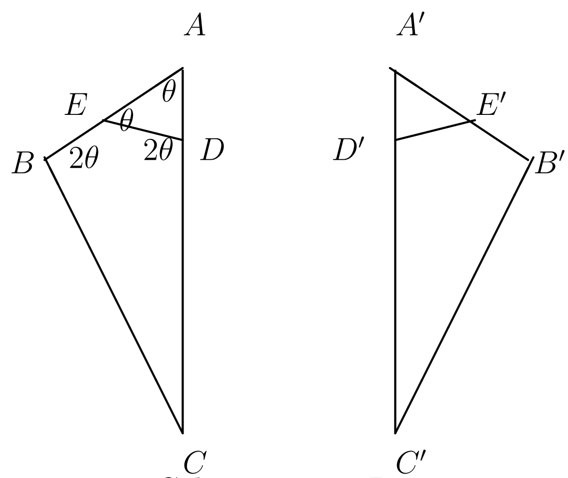
(b) Show that it is possible if one angle of the triangle is obtuse and twice as large as another.

Solution: [Wilson Ong, Y3, UWA] Denote the triangle by ABC and let θ be the angle at A.



Cake Box

(a) Let the angle at B be 3θ. Let D be the point on AC such that $\angle ABD = \theta$. Then the triangle ABD is isosceles with $AD = BD$. Also, $\angle BDC = \angle DBC = 2\theta$, so triangle BCD is isosceles with $BC = DC$.
Cut off triangle ABD and translate and rotate it so that A moves to D' and D moves to B'.



Cake Box

Let the angle at B be 2θ. Let D be the point on AC such that $BC = DC$ and let E be the point on AB such that $\angle AED = \theta$. Then the triangle AED is isosceles with $AD = ED$. Also, $\angle BDE = \angle DBE$, so triangle BDE is isosceles with $BE = DE$.
Cut off triangle AED and translate and rotate it so that A moves to E' and E moves to A'.

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10. Centre of mass

The centre of mass of a can of Coke (which may be idealised as a cylinder of height $2h$ and radius a and flat ends) is h above the base when the can is full or empty. Suppose the can is partly full. Show that the centre of mass is lowest when it lies in the surface of the liquid.

Does this result change when the can is replaced by a bottle of traditional Coke shape?

Solution: [Adrian Dudek, Y3, UWA] Let $C(y)$ be the height of the centre of mass when the height of the liquid is y, $0 \leq y \leq 2h$. Then $C(0) = h$, $C(2h) = h$.

When the liquid is removed from a full can, the centre of mass falls; likewise when liquid is added to an empty can. Since $C(y)$ is a continuous function, by Rolle's theorem, there is a value of y in $0 \leq y \leq 2h$ where $C'(y) = 0$. That is, there is a minimum value of $C(y)$ which is less than h. Thus $C(y) \leq h$ for $0 \leq y \leq 2h$ so by the intermediate value theorem there exists a value y for which $C(y) = y$; that is, the centre of mass lies in the surface.

When the centre of mass lies in the surface, if liquid is added, the height of the centre of mass increases, since adding mass above the existing centre of mass must raise the centre of mass.

Likewise, when the centre of mass lies in the surface, if liquid is removed, then the height of the centre of mass increases, since removing mass below the existing centre of mass must raise the centre of mass.

Hence the centre of mass is a minimum when it lies in the surface.

For a Coke bottle shape, the same argument applies although the height of the centre of mass of a full bottle is different from that of an empty bottle.

Alternative Solution

Suppose the shape of of the container is given by rotation of the curve $x = f(y)$ about the y–axis. (For the cylindrical can, $f(y) = a$.) Let the mass of the container be M and its empty centre of mass at height h.

The contribution of the container to the first moment of mass about the base is Mh.

The contribution to the first moment of the layer of fluid of thickness dy at height y is $y\rho\pi f^2(y)\,dy$, where ρ is the density, and the total contribution of the liquid is the integral of this.

Hence the centre of mass is

$$C(y) = \frac{\int_0^y u\rho\pi f^2(u)\,du + Mh}{\int_0^y \rho\pi f^2(u)\,du + M} = \frac{\int_0^y uf^2(u)\,du + Kh}{\int_0^y f^2(u)\,du + K}$$

where $M = \rho\pi K$.

$C'(y) = 0$ when $\left[\int_0^y f^2(u)\,du + K\right][yf^2(y)] = \left[\int_0^y uf^2(u)\,du + Kh\right][f^2(y)]$.

Since $f^2(y) \neq 0,\ y\int_0^y f^2(u)\,du + Ky = \int_0^y uf^2(u)\,du + Kh$.

Denote the solution of this equation by y^*, where

$$y^* = \frac{\int_0^{y^*} uf^2(u)\,du + Kh}{\int_0^{y^*} f^2(u)\,du + K} = C(y^*)$$

which says that the height of the surface and the minimum height of the centre of mass coincide.