The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2010 Problems with Solutions

1. Matrices of order 2.

Prove that for every integer $n \geq 2$ there are infinitely many $n \times n$ matrices with integer entries that are their own inverse.

Solution. [**By Jimmi Phangestu, 2nd year, Curtin**]
We claim that an $n \times n$ matrix, $A = (a_{ij})$, i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

with integer entries, is its own inverse, if it satisfies the following conditions:

(i) Except for possibly one entry, all non-diagonal entries a_{ij} are zero.

(ii) Main diagonal entries are 1 or -1, i.e. $a_{ii} = \pm 1$, for $1 \leq i \leq n$.

(iii) If in (i), non-diagonal entry $a_{ij} \neq 0$, then $a_{ii} = -a_{jj}$.

We now prove this claim. Suppose A satisfies (i), (ii), (iii), and let $B = A^2$. We must show $B = I$. We know

$$\begin{aligned} b_{ij} &= a_{i1}a_{1j} + a_{i2}a_{2j} + \cdots + a_{in}a_{nj} \\ &= \sum_{k=1}^{n} a_{ik}a_{kj}, \quad \text{for } i, j = 1, 2, \ldots, n. \end{aligned}$$

- For $i = j$,

$$\begin{aligned} b_{ii} = \sum_{k=1}^{n} a_{ik}a_{ki} &= a_{ii}a_{ii} + \sum_{\substack{k=1 \\ k\neq i}}^{n} a_{ik}a_{ki} \\ &= 1 + \sum_{\substack{k=1 \\ k\neq i}} 0, \qquad \text{since by (ii), } a_{ii} = \pm 1\text{, and by (i) at most one of } a_{ik}, a_{ki} \text{ is non-zero, for } k \neq i \\ &= 1. \end{aligned}$$

- For $i \neq j$,

$$b_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = a_{ii}a_{ij} + a_{ij}a_{jj} + \sum_{\substack{k=1 \\ k\neq i, k\neq j}}^{n} a_{ik}a_{kj}.$$

Again, by (i), at most one of a_{ik}, a_{kj} is non-zero, for $k \neq i$ and $k \neq j$, so that

$$\sum_{\substack{k=1 \\ k\neq i, k\neq j}}^{n} a_{ik}a_{kj} = 0.$$

Also,

$$\begin{aligned} a_{ii}a_{ij} + a_{ij}a_{jj} &= (a_{ii} + a_{jj})a_{ij} \\ &= 0, \end{aligned}$$

since either $a_{ij} = 0$, or by (iii), if $a_{ij} \neq 0$, then $a_{ii} = -a_{jj}$, so that $a_{ii} + a_{jj} = 0$. Thus,

$$b_{ij} = 0,\ i \neq j.$$

Thus $B = I$, and hence A is its own inverse as claimed.
Since one non-diagonal entry of A can be any non-zero integer, there are infinitely many $n \times n$ matrices A, with integer entries for any $n \geq 2$, such that A is its own inverse.
Method 2: It suffices to consider $n = 2$ because for any larger n we could take a 2×2 matrix in the top left corner, fill up the diagonal with 1s and all other entries with 0s.
For any integer b,

$$M = \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$$

satisfies $M^2 = I_2$ so is its own inverse.
Observe that this method is just a particular case of the first method.

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2. A binary operation.

Consider a binary operation $*$ on a set S, that is, for all $a, b \in S$, $a * b$ is in S.
Prove that if for all $a, b \in S$, $(a * b) * a = b$, then for all $a, b \in S$, $a * (b * a) = b$.

Solution. [**By Ferris Xu, 1st year, Curtin, and Yita Choong, 3rd year, UWA**]
Since $(a * b) * a = b$ for all $a, b \in S$, it follows that (for all $a, b \in S$)

$$(b * a) * b = a \qquad (1)$$
$$(c * b) * c = b, \quad \text{where } c = b * a. \qquad (2)$$

So, for all $a,\ b \in S$,

$$\begin{aligned} a * (b * a) &= \big((b * a) * b\big) * (b * a), && \text{by (1)} \\ &= b, && \text{by (2).} \end{aligned}$$

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3. Arithmetic equations.

Consider a system of m linear equations ($m > 1$) with real coefficients,

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i,\ i = 1, \ldots, m,$$

with a unique solution, such that

$$a_{11}, a_{12}, \ldots, a_{1n}, b_1, a_{21}, a_{22}, \ldots, a_{2n}, b_2, \ldots, a_{m1}, a_{m2}, \ldots, a_{mn}, b_m$$

is a non-trivial arithmetic progression (i.e. the common difference is not zero, so that the coefficients are not all equal).
What is that unique solution?

Solution. **[Based on solution by Ferris Xu, 1st year, Curtin]**
Let A be the system coefficient matrix, $B = [b_1\ b_2\ \dots\ b_m]^\top$, $a = a_{11}$ and k be the common difference of the coefficients' arithmetic progression, and to save some space, let $N = n + 1$. Writing the system of equations in matrix form and applying elementary row operations, we have:

$$
\begin{aligned}
&[A \mid B] \\
&= \left[\begin{array}{cccc|c}
a & a+k & \cdots & a+(n-1)k & a+nk \\
a+Nk & a+(N+1)k & \cdots & a+(N+n-1)k & a+(N+n)k \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
a+N(i-1)k & a+\big(N(i-1)+1\big)k & \cdots & a+\big(N(i-1)+(n-1)\big)k & a+\big(N(i-1)+n\big)k \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
a+N(m-1)k & a+\big(N(m-1)+1\big)k & \cdots & a+\big(N(m-1)+(n-1)\big)k & a+\big(N(m-1)+n\big)k
\end{array}\right] \\
&\sim \left[\begin{array}{cccc|c}
a & a+k & \cdots & a+(n-1)k & a+nk \\
Nk & Nk & \cdots & Nk & Nk \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
N(i-1)k & N(i-1)k & \cdots & N(i-1)k & N(i-1)k \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
N(m-1)k & N(m-1)k & \cdots & N(m-1)k & N(m-1)k
\end{array}\right], \quad R_j \leftarrow R_j - R_1,\ j = 2, \dots, m. \\
&\sim \left[\begin{array}{cccc|c}
a & a+k & \cdots & a+(n-1)k & a+nk \\
Nk & Nk & \cdots & Nk & Nk \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{array}\right], \quad R_j \leftarrow R_j - (j-1)R_2,\ j = 3, \dots, m. \\
&\sim \left[\begin{array}{cccc|c}
a & a+k & \cdots & a+(n-1)k & a+nk \\
1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{array}\right], \quad R_2 \leftarrow R_2/(Nk), \text{ (noting that } Nk \neq 0). \\
&\sim \left[\begin{array}{cccc|c}
1 & 1 & \cdots & 1 & 1 \\
a & a+k & \cdots & a+(n-1)k & a+nk \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{array}\right], \quad R_1 \leftrightarrow R_2. \\
&\sim \left[\begin{array}{cccc|c}
1 & 1 & \cdots & 1 & 1 \\
0 & k & \cdots & (n-1)k & nk \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{array}\right], \quad R_2 \leftarrow R_2 - aR_1. \\
&\sim \left[\begin{array}{cccc|c}
1 & 1 & \cdots & 1 & 1 \\
0 & 1 & \cdots & (n-1) & n \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{array}\right], \quad R_2 \leftarrow R_2/k, \text{ (noting that } k \neq 0).
\end{aligned}
$$

The final matrix above is in Row Echelon Form. So $\text{rank}(A) = \text{rank}(A|B) = 2$. For a unique solution, $n = 2$, in which case the Row Echelon matrix reduces further as follows:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right], \qquad R_1 \leftarrow R_1 - R_2.$$

The last matrix is in Reduced Row Echelon Form, and so we can read off the unique solution: $x_1 = -1,\ x_2 = 2$.

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4. Triples.

What are all the triples (x, y, z) of positive real numbers such that $x^{y/z} = y^{z/x} = z^{x/y}$?

Solution. Firstly, suppose $x^{y/z} = y^{z/x} = z^{x/y}$. Then

$$\left(\frac{1}{x}\right)^{y/z} = \left(\frac{1}{y}\right)^{z/x} = \left(\frac{1}{z}\right)^{x/y}$$
$$\implies \left(\frac{1}{x}\right)^{\frac{1/z}{1/y}} = \left(\frac{1}{y}\right)^{\frac{1/x}{1/z}} = \left(\frac{1}{z}\right)^{\frac{1/y}{1/x}}.$$

Thus, if (x, y, z) is a solution then so is $(1/x, 1/z, 1/y)$.

Now note that, for $a, b > 0$,

$$a^b < 1 \quad \text{if and only if} \quad a < 1.$$

Consequently, if one of x, y, z is less than 1, then they are all less than 1. Consider the case where $x, y, z \geq 1$. Since the equations have rotational symmetry, we may assume without loss of generality that $x \leq z$ and $y \leq z$, whence

$$\begin{aligned} &\frac{y}{z} \leq 1 \text{ and } \frac{z}{x} \geq 1 \\ &\therefore x \geq x^{y/z} = y^{z/x} \geq y \\ &\therefore \frac{x}{y} \geq 1 \\ &\therefore x \geq x^{y/z} = z^{x/y} \geq z \\ &\therefore x = z, \qquad \text{since we had assumed } x \leq z \\ &\therefore x^{y/x} = y = x^{x/y}. \end{aligned} \tag{3}$$

Now, if $x = 1$ then (3) reduces to $1 = x = y$. On the other hand, if $x > 1$ then $x^{y/x} = x^{x/y}$ gives $y/x = x/y$ and hence again $x = y$, since $x, y > 0$. Either way, we have $x = y = z$.

If instead $x, y, z \leq 1$, then we apply the above argument to $1/x, 1/z, 1/y$ and so deduce $1/x = 1/z = 1/y$, which again implies $x = y = z$.
Checking we see that $x = y = z$ is indeed a solution, and hence all the positive real triples (x, y, z) satisfying $x^{y/z} = y^{z/x} = z^{x/y}$ are

$$(x, y, z) \in \{(t, t, t) \mid t > 0\}.$$

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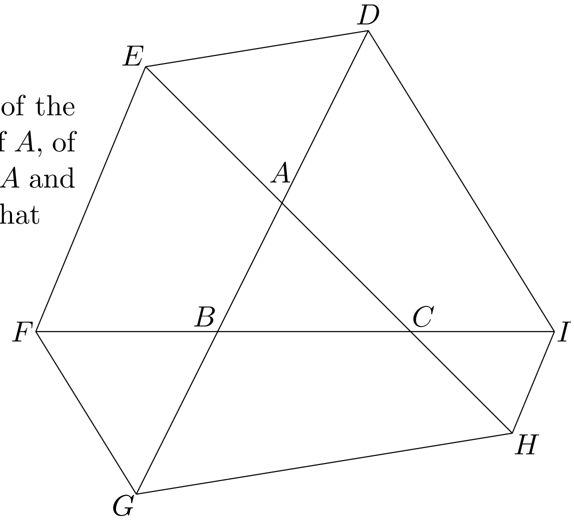
5. Triangle in a circle.

Let ABC be a triangle in the plane. Extend the sides AB and AC on the other side of A with segments of size $|BC|$. Similarly, extend the sides BA and BC on the other side of B with segments of size $|AC|$ and the sides CA and CB on the other side of C with segments of size $|AB|$.

Prove that the resulting six points outside the triangle are concyclic.

Solution. [**By Ferris Xu, 1st year, Curtin, and Yita Choong, 3rd year, UWA**]
Let $\alpha = \angle BAC$, $\beta = \angle CBA$, $\gamma = \angle ACB$. Then

$$\alpha + \beta + \gamma = 180°.$$



Let D, E, F, G, H and I be the endpoints of the extensions of AB and AC on the other side of A, of BC and BA on the other side of B, and of CA and CB on the other side of C, respectively, so that

$$\begin{aligned} AE = AD &= BC, \\ BF = BG &= AC \text{ and} \\ CH = CI &= AB. \end{aligned}$$

Since $AE = AD$,

$$\begin{aligned} &\triangle EAD \text{ is isosceles} \\ &\angle EAD = \angle BAC = \alpha, \quad \text{(vertically opposite angles)} \\ \therefore\ &\angle AED = \angle ADE = 90° - \alpha/2 \\ &AG = AB + BG \\ &\quad\ = CH + AC = AH \\ \therefore\ &\triangle GAH \text{ is isosceles, and } \angle A = \alpha \\ \therefore\ &\angle AGH = \angle AHG = 90° - \alpha/2. \end{aligned}$$

Similarly,

$$\begin{aligned} \angle BFG = \angle BGF = \angle BDI = \angle BID &= 90° - \beta/2 \\ \text{and } \angle CIH = \angle CHI = \angle CEF = \angle CFE &= 90° - \gamma/2 \\ \therefore \angle EDI + \angle EFI &= 90° - \alpha/2 + 90° - \beta/2 + 90° - \gamma/2 \\ &= 270° - 180°/2 \\ &= 180°. \end{aligned}$$

Similarly, $\angle HED + \angle HID = 180°$ and $\angle DGH + \angle DIH = 180°$.
$\therefore$ $EFID$, $EHID$ and $GHID$ are cyclic.
Since three noncollinear points are sufficient to determine a circle, $EFID$, $EHID$ and $GHID$ (each sharing 3 points with the next) have the same circumcircle.
Therefore the six points D, E, F, G, H and I (outside $\triangle ABC$) lie on the same circle, i.e. these six points are concyclic.

6. A happy class.

A number of students sit in a circle while their teacher hands out Mars bars. Each student initially has an even number of Mars bars, but not necessarily the same number. When the teacher blows a whistle, each student simultaneously gives half of his or her Mars bars to the neighbour on their right. Any student who now has an odd number of Mars bars eats one.

Show that after finitely many iterations of this procedure, all students have the same number of Mars bars.

Solution. Let N be the number of students. Call the Mars bars possessed by a student, a *pile*. We first establish some invariant properties.

Take some point in time, prior to the whistle being blown (i.e. a time when Mars bars are not being consumed, so that each pile has an even number of Mars bars). Let m and M be the minimum and maximum numbers of Mars bars in a pile.

Since each pile has an even number of Mars bars prior to the whistle being blown, each student passes an integer number of Mars bars to their right. Consider any student possessing m Mars bars. At the whistle, each such student gives $m/2$ Mars bars to their right neighbour and receives at least $m/2$ from their left, and so then has either m Mars bars which remains m (because m is even) or has at least $m+1$ Mars bars which after the consumption of at most one Mars bar, is still at least m.

So after an iteration the minimum pile size cannot fall below m.

Similarly, the maximum pile size cannot rise above M.

So the pile sizes remain bounded after the whistle, above and below by M and m, respectively.

Also observe that a pile having size less than M before the whistle necessarily has size less than M after the whistle. Hence a pile size cannot increase *to* M.

Now we show that after finitely many iterations of this procedure, all students have the same number of Mars bars, which we will call the *terminal state*.

If $M = m$ then all pile sizes are equal and we are at the terminal state. Suppose $M \neq m$, so that at least one student has fewer than M Mars bars. Consider a sequence of students sitting next to one another whose piles are of size M bordered by students with fewer than M Mars bars. Since not all the students have M Mars bars, each student with M Mars bars is in such a sequence. Suppose the sequence has k students. After the whistle the leftmost student of the sequence will have a pile of size less than M, but each of the rest of the students will give away as many Mars bars as they receive, and so remain with M Mars bars, i.e. after the whistle a sequence of k consecutive students with M Mars bars becomes a sequence of $k-1$ students with M Mars bars.

If K is the length of the longest sequence of students with M Mars bars, then after K iterations all students will have fewer than M Mars bars, and note that K is necessarily less than N (the number of students). At this stage we will have a new maximum bound of at most $M-2$ (since it must be even) and a minimum pile size that is at least m.

Call the new actual maximum and minimum M and m, respectively, and repeat the argument. Since by this process the maximum strictly decreases and the minimum does not decrease, we must end with the maximum and minimum being equal, in which case, we have reached the terminal state.

If the initial maximum and minimum pile sizes are M_0 and m_0, respectively, then the number of iterations it takes before the maximum and minimum are equal can be no more than $N(M_0 - m_0)/2$, the bound of the number of iterations it takes for a change in (M, m), times a bound on the number of changes of (M, m), and this is certainly finite.

Thus after finitely many iterations of this procedure, all students have the same number of Mars bars.

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7. Steady permutations.

A permutation π of $\{1, 2, \ldots, n\}$ is *steady* if $\pi(i+1) - \pi(i) \leq 1$ for all $i \in \{1, 2, \ldots, n-1\}$. How many steady permutations of $\{1, 2, \ldots, n\}$ are there?

Solution. [**By Ferris Xu, 1st year, Curtin**]
First, let's agree on a notation, and represent a permutation π by its list of images of $1, 2, \ldots, n$, in that order, i.e.

$$\pi = [\pi(1), \pi(2), \ldots, \pi(n)],$$

e.g. for $n = 3$, $\pi = [2, 3, 1]$ says $\pi(1) = 2$, $\pi(2) = 3$ and $\pi(3) = 1$.
With this notation, we can recognise a *steady* permutation as one for which the component entries either increase by no more than one or decrease, as we proceed from left to right. So, $\pi = [2, 3, 1]$ is steady.

Let $f(n)$ denote the number of steady permutations of $\{1, 2, \ldots, n\}$.
We will prove $f(n) = 2^{n-1}$ by induction.
For $n = 1$, $[1]$ is the only permutation and it is steady, i.e. $f(1) = 1 = 2^{1-1}$.
For $n = 2$, $[1, 2]$ and $[2, 1]$ are the only permutations and both are steady, i.e. $f(2) = 2 = 2^{2-1}$.
Now, assume $f(k) = 2^{k-1}$ and consider a steady permutation of length $k + 1$.
Let a be the element of $\{1, 2, \ldots, k+1\}$ such that $\pi(a) = k + 1$.
If $a = 1$, then $k + 1$ is the first element of π. Removing $k + 1$ produces a steady permutation of length k, by the recognition criterion mentioned above.
If $a \neq 1$. Then

$$\begin{aligned} 1 &\geq \pi(a) - \pi(a-1) \\ &= k + 1 - \pi(a-1) \\ \therefore \pi(a-1) &\geq k. \end{aligned}$$

So $\pi(a-1) = k$, since $k + 1$ the only other option is already taken.
Thus $\pi = [\ldots, k, k{+}1, \pi(a{+}1), \ldots]$, with $\pi(a{+}1) < k = \pi(a{-}1)$, so that $\pi(a{+}1){-}\pi(a{-}1) \leq 1$, and hence if $\pi(a) = k + 1$ is removed the resulting permutation of length k is still steady.
Therefore all steady permutations of length $k + 1$ are the result of adding $k + 1$ to a steady permutation of length k.
Also, by the above argument, there are exactly two ways to create a steady permutation of length $k + 1$ from a steady permutation π of length k:

insert $k + 1$ at the beginning of the k-tuple for π, or
insert $k + 1$ as the element immediately after k.

$\therefore f(k+1) = 2f(k) = 2 \cdot 2^{k-1} = 2^k = 2^{k+1-1}$.
Hence by the Principle of Mathematical Induction, $f(n) = 2^{n-1}$ for all natural numbers n, i.e. there are 2^{n-1} steady permutations of $\{1, 2, \ldots, n\}$.

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8. Pretty polynomials.

A polynomial $p(x)$ is *pretty* if, for any given point (x_0, y_0) of the plane $\mathbb{R}^2$, one can find a tangent to the curve $y = p(x)$ going through (x_0, y_0).

For which n are all polynomials of degree n pretty?

Solution. [**By Aedan Pope, 3rd year, UWA**]
First observe that for polynomials $p(x)$ of degree ≤ 1, $y = p(x)$ coincides with its tangent at every point. In particular, such polynomials $p(x)$ are of form

$$p(x) = c_1 x + c_0,$$

for some $c_0, c_1 \in \mathbb{R}$ (with $c_1 = 0$ if the degree < 1), and while $(0, c_0)$ lies on $y = p(x)$, $(0, c_0+1)$ does not. Thus no polynomial $p(x)$ of degree ≤ 1 is pretty.

Consider the tangent ℓ to $y = p(x)$ at $x = a$, $a \in \mathbb{R}$. The point $\big(a, p(a)\big)$ is on ℓ, and ℓ has gradient $p'(a)$. Thus ℓ is given by

$$y - p(a) = p'(a)(x - a).$$

For a given point $(x_0, y_0) \in \mathbb{R}^2$ to lie on ℓ,

$$y_0 - p(a) = p'(a)(x_0 - a),$$

or equivalently,

$$p(a) - ap'(a) + x_0 p'(a) - y_0 = 0 \tag{4}$$

must hold.

Consider $p(x) = x^{2m}$, $m \in \mathbb{N}$ and $(x_0, y_0) = (0, 1)$. Then the left hand side of (4) reduces to

$$a^{2m} - a \cdot 2m \cdot a^{2m-1} - 1 = -(2m-1)a^{2m} - 1 \leq -1,$$

so that (4) cannot be satisfied for any $a \in \mathbb{R}$, i.e. $p(x)$ is not pretty. Thus, for each even degree there exist polynomials $p(x)$ that are not pretty.

What's left to consider are polynomials $p(x)$ of odd degree $n \geq 3$. For such $p(x)$ to be pretty, there must exist an $a \in \mathbb{R}$, for each $(x_0, y_0) \in \mathbb{R}^2$, such that (4) holds. Consider the polynomial q in a formed by the left hand side of (4),

$$q(a) := p(a) - ap'(a) + x_0 p'(a) - y_0.$$

If the leading coefficient of p is c_n, then the leading coefficient of q is

$$c_n - nc_n = (1 - n)c_n \neq 0.$$

So q has the same (odd) degree as p, and hence has range $\mathbb{R}$, and hence, by the Intermediate Value Theorem, there exists $a \in \mathbb{R}$, such that $q(a) = 0$. Thus (4) is satisfied for some $a \in \mathbb{R}$, for each $(x_0, y_0) \in \mathbb{R}^2$, and so all polynomials $p(x)$ of odd degree $n \geq 3$ are pretty.

Thus, only for odd $n \geq 3$ are all polynomials of degree n pretty.

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9. Alice has a birthday.

Alice: "Today is my birthday and my age is a root of this polynomial in x with integer coefficients."
Bob: "If I replace x by 7, I get 77."
Alice: "Do I look like I am 7?"
Bob: "No, indeed. I now replace x by a bigger integer N and I get 85, still not 0."
Alice: "Isn't it obvious that I am older than N?"

How old is Alice?

Solution. [**By Ferris Xu, 1st year, Curtin, and Aedan Pope, 3rd year, UWA**]
We will use the following lemma a number of times.

Lemma. *If $a, b \in \mathbb{Z}$ and $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ is a polynomial with coefficients $c_n, c_{n-1}, \ldots, c_0 \in \mathbb{Z}$ then*

$$(a-b) \,\big|\, \big(p(a) - p(b)\big).$$

Proof. We have,

$$\begin{aligned} p(a) - p(b) &= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_0 - (c_n b^n + c_{n-1} b^{n-1} + \cdots + c_0) \\ &= c_n(a^n - b^n) + c_{n-1}(a^{n-1} - b^{n-1}) + \cdots + c_1(a-b). \end{aligned}$$

Thus, since $a-b$ divides $a^k - b^k$ for any integer $k \geq 1$, it follows that $a-b$ divides $p(a)-p(b)$. □

Let $p(x)$ be the polynomial with integer coefficients, and let $A \in \mathbb{Z}$ be Alice's age. Then we have

$$p(7) = 77,\ p(N) = 85,\ p(A) = 0,$$

with $7 < N < A$. Now, $N - 7 > 0$ and by the lemma,

$$\begin{aligned} (N-7) \,|\, 8 &= p(N) - p(7) \\ \implies N - 7 &\in \{1, 2, 4, 8\} \\ \implies N &\in \{8, 9, 11, 15\} \end{aligned} \tag{5}$$

Also, $A - 7 > 0$ and again by the lemma,

$$\begin{aligned} (A-7) \,|\, {-77} &= p(A) - p(7) \\ \implies A - 7 &\in \{1, 7, 11, 77\} \\ \implies A &\in \{8, 14, 18, 84\} \end{aligned} \tag{6}$$

Lastly, $A - N > 0$ and by the lemma one last time,

$$\begin{aligned} (A-N) \,|\, {-85} &= p(A) - p(N) \\ \implies A - N &\in \{1, 5, 17, 85\} \end{aligned} \tag{7}$$

Using (5) and (6) we create a table of possibilities for $A - N$:

$N \backslash A$	8	14	18	84
8	0	6	10	76
9	−1	5	9	75
11	−3	3	7	73
15	−7	−1	3	69

Table 1: Values of $A - N$ given A and N

Comparing the values of $A - N$ in the body of Table 1 with the possibilities in (7), we see agreement only in the case $A - N = 5$, in which case $A = 14$ and $N = 9$.

So Alice is 14.

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10. Unavoidable numbers.

A real number $r \in (0, 1]$ is *unavoidable* if for any continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(0) = f(1) = 0$, the graph of f has an horizontal chord of length r (in other words, there exists x in $[0, 1 - r]$ such that $f(x) = f(x + r)$).

(a) Prove that all numbers of the form $1/n$ (n a positive integer) are unavoidable.

(b) Prove that real numbers $r \in (0, 1)$ that are *not* of the form $1/n$ (n a positive integer) are *not* unavoidable.

Solution. [**Partly based on solution by Robert Palmer, 3rd year, UWA**]

(a) Take a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(0) = f(1) = 0$, and $n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{i=1}^{n}\Big(f\Big(\frac{i}{n}\Big) - f\Big(\frac{i-1}{n}\Big)\Big) \qquad (8)\\ &= \Big(f\Big(\frac{1}{n}\Big) - f\Big(\frac{0}{n}\Big)\Big) + \Big(f\Big(\frac{2}{n}\Big) - f\Big(\frac{1}{n}\Big)\Big) + \cdots + \Big(f\Big(\frac{n}{n}\Big) - f\Big(\frac{n-1}{n}\Big)\Big)\\ &= f(1) - f(0) = 0.\end{aligned}$$

Case 1: One of the terms of the summation (8), the k^{th} say, is zero. Then for $x = \dfrac{k-1}{n}$, $f(x) = f\big(x + \frac{1}{n}\big) = f\big(\frac{k}{n}\big)$, so that the graph of f has a horizontal chord of length $\frac{1}{n}$.

Case 2: None of the terms of (8) is zero. Define the function $g : \big[0, 1 - \frac{1}{n}\big] \to \mathbb{R}$ by

$$g(x) = f\Big(x + \frac{1}{n}\Big) - f(x).$$

Since f is continuous on $[0, 1]$, it follows that g is continuous on $\big[0, 1 - \frac{1}{n}\big]$. Writing (8) in terms of g, we have:

$$g\Big(\frac{0}{n}\Big) + g\Big(\frac{1}{n}\Big) + \cdots + g\Big(\frac{n-1}{n}\Big) = 0, \qquad (9)$$

and since none of the terms of (9) is zero, but their sum is zero, some of its terms are positive and some are negative. In particular, there exists $\ell \in \{0, 1, \dots, n-2\}$ such that $g\big(\frac{\ell}{n}\big)$ and $g\big(\frac{\ell+1}{n}\big)$ are of opposite sign. Since g is continuous on $\big[0, 1 - \frac{1}{n}\big]$ and hence continuous on $\big[\frac{\ell}{n}, \frac{\ell+1}{n}\big]$, by the Intermediate Value Theorem there exists $c \in \big(\frac{\ell}{n}, \frac{\ell+1}{n}\big) \subseteq \big(0, 1 - \frac{1}{n}\big)$ such that $g(c) = 0$, i.e. for $x = c$ we have $f(x) = f\big(x + \frac{1}{n}\big)$, so that again the graph of f has a horizontal chord of length $\frac{1}{n}$.

Thus in either case, we have that $\frac{1}{n}$ is unavoidable for each positive integer n.

(b) Take $r \in (0,1) \setminus \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$, let $s = \frac{1}{r}$, and let $f : [0,1] \to \mathbb{R}$ such that

$$f(x) = x\sin^2(s\pi) - \sin^2(s\pi x).$$

Then f is continuous on $[0,1]$ and $f(0) = f(1) = 0$. However, for all $x \in \left[0, 1 - \frac{1}{s}\right]$,

$$\begin{aligned} f\left(x + \frac{1}{s}\right) - f(x) &= \left(x + \frac{1}{s}\right)\sin^2(s\pi) - \sin^2\left(s\pi\left(x + \frac{1}{s}\right)\right) - \left(x\sin^2(s\pi) - \sin^2(s\pi x)\right) \\ &= \frac{1}{s}\sin^2(s\pi) - \sin^2(s\pi x + \pi) + \sin^2(s\pi x) \\ &= \frac{1}{s}\sin^2(s\pi) \end{aligned}$$

which is constant on $\left[0, 1 - \frac{1}{s}\right]$, but non-zero, since the only zeros of $\sin x$ are integer multiples of π and s is not an integer.
Thus for each $r \in (0,1) \setminus \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$, there exists a function f satisfying all the required conditions that does not have a horizontal chord of length r.
Hence, the real numbers $r \in (0,1)$ that are not of the form $\frac{1}{n}$ (n a positive integer) are not unavoidable.