The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2011 Problems with Solutions

1. Argument duplication

Determine all real polynomials $P(x)$ such that $P(2x) = P'(x) \cdot P''(x)$ for all $x \in \mathbb{R}$.

> **Solution.** [**By Aaron Maynard and Phillip Meng, both 3rd year, UWA**]
> Let $P(x)$ be a real polynomial such that $P(2x) = P'(x) \cdot P''(x)$ for all $x \in \mathbb{R}$.
> The zero polynomial is one such polynomial $P(x)$.
> Suppose that $P(x)$ is not the zero polynomial and let n be the degree of $P(x)$. Then the degree of $P(2x)$ is n. If n is equal to 0 or 1, then $P'(x) \cdot P''(x)$ will be the zero polynomial, and so will not be equal to $P(2x)$. If $n \geq 2$, then the degree of $P'(x) \cdot P''(x)$ will be $(n-1) + (n-2) = 2n - 3$. Therefore, $P(2x) = P'(x) \cdot P''(x)$ implies that $n = 2n - 3$ and hence $n = 3$. So, $P(x)$ is a cubic polynomial of the form $a + bx + cx^2 + dx^3$, with $d \neq 0$.
> Therefore,
>
> $$P(2x) = a + 2bx + 4cx^2 + 8dx^3$$
> $$P'(x) \cdot P''(x) = (b + 2cx + 3dx^2)(2c + 6dx)$$
> $$= 2bc + (4c^2 + 6bd)x + 18cdx^2 + 18d^2x^3$$
>
> and so
> $$a + 2bx + 4cx^2 + 8dx^3 = 2bc + (4c^2 + 6bd)x + 18cdx^2 + 18d^2x^3$$
>
> which implies that
>
> $$a = 2bc$$
> $$2b = 4c^2 + 6bd$$
> $$4c = 18cd$$
> $$8d = 18d^2.$$
>
> Solving this set of simultaneous equations, first we find $d = \frac{4}{9}$ (recall $d \neq 0$), from which we see that successively $c = 0$, $b = 0$ and $a = 0$, giving us the solution $P(x) = \frac{4}{9}x^3$.
> Therefore, the only real polynomials $P(x)$ such that $P(2x) = P'(x) \cdot P''(x)$ for all $x \in \mathbb{R}$ are $P(x) = 0$ and $P(x) = \frac{4}{9}x^3$.

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2. So tyred

A car has 4 tyres, and in its boot are stored 3 spare tyres. Each tyre can be used for $40\,000$ km.

What is the maximum distance that the car can be driven? (The tyres can be interchanged as many times as you want.)

Solution. [**By Aaron Maynard and Phillip Meng, both 3rd year, UWA**]
There are 7 tyres, each of which can travel $40\,000$ km, giving a total of $280\,000$ tyre km. Since 4 tyres need to be attached to the car for it to travel, the maximum distance the car can travel is at most $\frac{280\,000}{4} = 70\,000$ km.
The table below shows one way the car may be driven for $70\,000$ km:

Distance car has travelled

	First 30 000 km	Next 10 000 km	Next 10 000 km	Next 10 000 km	Next 10 000 km
Tyre 1	✓	✓			
Tyre 2	✓		✓		
Tyre 3	✓			✓	
Tyre 4	✓				✓
Tyre 5		✓	✓	✓	✓
Tyre 6		✓	✓	✓	✓
Tyre 7		✓	✓	✓	✓

Thus we have shown that the maximum distance that the car can travel is bounded above by $70\,000$ km, and that $70\,000$ km is achievable.
Therefore, the maximum distance that the car can travel is $70\,000$ km.

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3. Inequality

Prove that $(n!)^2 > n^n$ for all $n \geq 3$.

Solution. [**By Tuo Li, 2nd year, UWA**]
We can organise the factors of $(n!)^2$ in the following way:

$$(n!)^2 = (n \cdot 1) \cdot ((n-1) \cdot 2) \cdot ((n-2) \cdot 3) \cdots (1 \cdot n) = \prod_{i=0}^{n-1} (n-i)(i+1).$$

Hence the inequality $(n!)^2 > n^n$ can be rewritten as

$$\prod_{i=0}^{n-1} \frac{(n-i)(i+1)}{n} > 1. \tag{1}$$

For each integer $i \in [0, n-1]$, $\dfrac{(n-i)(i+1)}{n} \geq 1$, since

$$\frac{(n-i)(i+1)}{n} = 1 + \frac{i(n-i-1)}{n} \text{ and } i, n-i-1 \geq 0.$$

Moreover we have equality $\dfrac{(n-i)(i+1)}{n} = 1$ exactly when $i(n-i-1) = 0$, that is, when $i = 0$ or $i = n-1$.
Therefore we have

$$\prod_{i=0}^{n-1} \frac{(n-i)(i+1)}{n} \geq 1 \cdot 1 \cdots 1 = 1.$$

Since $n \geq 3$, the product has at least 3 factors; so at least one of the factors (say for $i = 1$) is strictly greater than 1. So, for $n \geq 3$ the inequality (1) is strict. Hence,

$$(n!)^2 > n^n \text{ for all } n \geq 3.$$

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4. Black or white cube

A $3 \times 3 \times 3$ cube is assembled from 27 $1 \times 1 \times 1$ cubes all of whose faces are white. We paint all of the faces of the large cube black, and then disassemble it. A blindfolded man reassembles the large cube from the 27 little cubes.

What is the probability that all the faces of the reassembled cube are completely black?

> **Solution.** [**By Aaron Maynard, 3rd year, UWA**]
> Of the 27 little cubes, one is completely white, six are black on one face, twelve are black on two (edge-sharing) faces and eight are black on three (corner-sharing) faces. When the large cube is reassembled, each of the little cubes has 27 possible positions and 24 possible orientations. Therefore, the total number of possible large cubes is
> $$27! \times 24^{27}.$$
> Suppose that all of the faces of the reassembled large cube are completely black. There is 1! possible way to position the white cube, 6! possible ways to position the cubes with one black face, 12! for the cubes with two black faces, and 8! for the cubes with three black faces. There are also 24 ways to orient the white cube, 4 ways to orient each of the cubes with one black face, 2 ways to orient each of the cubes with two black faces, and 3 ways to orient each of the cubes with three black faces. Therefore, the total number of possible large, completely black cubes is
> $$1! \times 6! \times 12! \times 8! \times 24^1 \times 4^6 \times 2^{12} \times 3^8.$$
> Therefore, the probability that the reassembled cube is completely black is
> $$\frac{6! \times 12! \times 8! \times 24 \times 4^6 \times 2^{12} \times 3^8}{27! \times 24^{27}} = \frac{6! \times 12! \times 8!}{27! \times 24^{18}}$$
> $$= \frac{1}{5\,465\,062\,811\,999\,459\,151\,238\,583\,897\,240\,371\,200}.$$
> This is approximately 1.829×10^{-37}; in other words, a very small probability.

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5. Touching circles

Let circles K_1 and K_2 be touching at point P, with the smaller circle K_1 inside K_2. Let line ℓ be tangent to K_1 at A and intersect K_2 at points B and C.

Show that PA is the (interior) angle bisector of $\angle BPC$.

> **Solution.** [**By Tuo Li, 2nd year, UWA**]
> We make extensive use of the following standard theorem.
>
> **Theorem 1** (Alternate Segment Theorem)**.** *If PQ is a chord of a circle K, X is a point external to K such that XP is the tangent line to K through P, and A is another point on K that is on the opposite side of PQ to X, then $\angle XPQ = \angle PAQ$.*
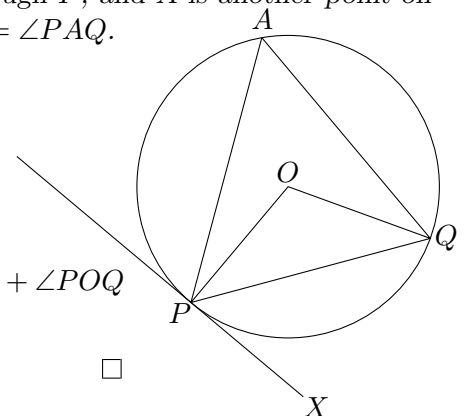>
> **Proof.** Let O be the centre of K. Then
> $$\angle PAQ = \tfrac{1}{2}\angle POQ, \quad \text{angles at circumference}$$
> $$\text{and centre standing on}$$
> $$\text{same arc } PQ$$
> $$\angle OPQ + \angle XPQ = 90° \ \text{ and } \angle OPQ = \angle OQP$$
> $$\therefore 180° = \angle OPQ + \angle OQP + \angle POQ = 2\angle OPQ + \angle POQ$$
> $$\therefore 90° = \angle OPQ + \tfrac{1}{2}\angle POQ$$
> $$\therefore \angle XPQ = \tfrac{1}{2}\angle POQ = \angle PAQ \qquad \square$$

Now let us consider the given problem.

Let ℓ intersect the tangent line to K_1 and K_2 (through P) at X.
Without loss of generality, take C to be farther than B from X.
Let BP intersect K_1 at D. Let PC intersect K_1 at E.
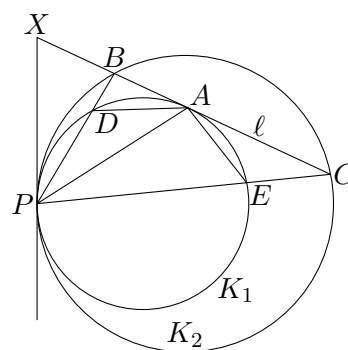By Theorem 1,

$$\angle DPA = \angle BAD, \quad \text{(chord } AD, \text{ tangent } BA, \text{ circle } K_1) \quad (2)$$

$$\angle EPA = \angle CAE, \quad \text{(chord } AE, \text{ tangent } CA, \text{ circle } K_1) \quad (3)$$

$$\angle DAP = \angle XPB, \quad \text{(chord } PD, \text{ tangent } XP, \text{ circle } K_1) \quad (4)$$

$$\angle BCP = \angle XPB, \quad \text{(chord } PB, \text{ tangent } XP, \text{ circle } K_2) \quad (5)$$

$$\angle AEP = \angle BAP, \quad \text{(chord } AP, \text{ tangent } BA, \text{ circle } K_1) \quad (6)$$

So we have

$$\angle BAD + \angle DAP = \angle BAP$$
$$= \angle AEP, \qquad \text{by (6)}$$
$$= \angle CAE + \angle ACE, \qquad \angle AEP \text{ is exterior to } \triangle EAC$$
$$\therefore \angle BAD = \angle CAE, \qquad \text{since } \angle DAP = \angle BCP, \text{ by (4) and (5),}$$
$$\text{and } \angle BCP = \angle ACE \text{ (same angle)}$$
$$\therefore \angle DPA = \angle EPA \qquad \text{by (2) and (3)}$$
$$\therefore PA \text{ is the angle bisector of } \angle DPE = \angle BPC.$$

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6. Integral boxes

A vector $\underset{\sim}{\boldsymbol{v}} = (x, y, z) \in \mathbb{R}^3$ is **integral** if each component is an integer.

Prove that if $\underset{\sim}{\boldsymbol{u}}$, $\underset{\sim}{\boldsymbol{v}}$ and $\underset{\sim}{\boldsymbol{w}}$ are mutually orthogonal integral vectors with the same length L, then L is an integer.

Solution. [**By Mitchell Misich, 3rd year, UWA**]
Given three mutually orthogonal integral vectors $\underset{\sim}{\boldsymbol{u}}, \underset{\sim}{\boldsymbol{v}}, \underset{\sim}{\boldsymbol{w}}$ with the same length $L \neq 0$, they define a parallelepiped that is actually a cube. (Note that we can ignore the trivial case $L = 0$, since $0 \in \mathbb{Z}$, and so there is nothing to prove in this case.)
The volume of the cube defined by $\underset{\sim}{\boldsymbol{u}}, \underset{\sim}{\boldsymbol{v}}, \underset{\sim}{\boldsymbol{w}}$ is $L^3 = |\underset{\sim}{\boldsymbol{u}} \cdot (\underset{\sim}{\boldsymbol{v}} \times \underset{\sim}{\boldsymbol{w}})|$ and so L^3 is an integer.
Furthermore, $L^2 = \underset{\sim}{\boldsymbol{u}} \cdot \underset{\sim}{\boldsymbol{u}}$ and so L^2 is an integer.
Therefore, $L = \dfrac{L^3}{L^2}$ is rational, and since L^2 is an integer, L must also be an integer.

Theorem. If $N \in \mathbb{Q}$ and $N^2 \in \mathbb{Z}$ then $N \in \mathbb{Z}$.

Proof. Since $N \in \mathbb{Q}$, $N = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$ and a, b coprime, i.e. $\gcd(a, b) = 1$.
Since $N^2 \in \mathbb{Z}$, we have b^2 divides a^2 and hence for any prime divisor p of b, we have

$$p^2 \,|\, a^2$$
$$\implies p \,|\, a^2 = a \cdot a$$
$$\implies p \,|\, a \text{ or } p \,|\, a, \qquad \text{by Euclid's Lemma}$$
$$\implies p \,|\, a.$$

Thus any prime divisor p of b is also a divisor of a, so that $\gcd(a, b) \geq p$.
So we have a contradiction, unless b has no prime divisors.
Therefore, $b = 1$ and $N \in \mathbb{Z}$. $\qquad\qquad \square$

7. Counting digits

From a positive integer n (in decimal form), we form another one $\alpha(n)$ as follows: write the number of even digits of n then the number of odd digits of n, then the total number of digits of n. For instance $\alpha(8484848486775) = 10313$ and $\alpha(7777) = 044 = 44$.

Is there a number k such that, for any n, $\alpha^i(n) = k$, for i sufficiently large?

> **Solution.** [**Inspired by Karl Beidatsch, 3rd year, Curtin**]
> The answer is: Yes, with number $k = 123$ being always ultimately reached.
> Call a sequence, $\quad n, \alpha(n), \alpha^2(n), \dots$ an α-*sequence*, and for brevity let us write it in the form
> $$n \overset{\alpha}{\longmapsto} \alpha(n) \overset{\alpha}{\longmapsto} \alpha^2(n) \overset{\alpha}{\longmapsto} \cdots.$$
> We first show that if n has 1, 2 or 3 digits, then $\alpha^i(n) = 123$ for i sufficiently large.
> (Note that whenever an α-sequence reaches a number that has been seen previously, we can stop, since the sequence thereafter will continue in the manner of the sequence previously analysed; we indicate this by a statement of form *"proceed as above"*.)
>
> If n has 1 digit which is odd, then: $\qquad n \overset{\alpha}{\longmapsto} 11 \overset{\alpha}{\longmapsto} 22 \overset{\alpha}{\longmapsto} 202 \overset{\alpha}{\longmapsto} 303 \overset{\alpha}{\longmapsto} 123.$
>
> If n has 1 digit which is even, then: $\qquad n \overset{\alpha}{\longmapsto} 101 \overset{\alpha}{\longmapsto} 123.$
>
> If n has 2 digits, both odd, then: $\qquad n \overset{\alpha}{\longmapsto} 22 \overset{\alpha}{\longmapsto}$ proceed as for 1 odd digit.
>
> If n has 2 digits, both even, then: $\qquad n \overset{\alpha}{\longmapsto} 202 \overset{\alpha}{\longmapsto}$ proceed as for 1 odd digit.
>
> If n has 2 digits, one odd, one even, then: $\quad n \overset{\alpha}{\longmapsto} 112 \overset{\alpha}{\longmapsto} 123.$
>
> If n has 3 digits, all odd, then: $\qquad n \overset{\alpha}{\longmapsto} 33 \overset{\alpha}{\longmapsto} 22 \overset{\alpha}{\longmapsto}$ proceed as for 1 odd digit.
>
> If n has 3 digits, two odd, one even, then: $\quad n \overset{\alpha}{\longmapsto} 123.$
>
> If n has 3 digits, one odd, two even, then: $\quad n \overset{\alpha}{\longmapsto} 213 \overset{\alpha}{\longmapsto} 123.$
>
> If n has 3 digits, all even, then: $\qquad n \overset{\alpha}{\longmapsto} 303 \overset{\alpha}{\longmapsto}$ proceed as for 1 odd digit.

In particular, $123 \overset{\alpha}{\longmapsto} 123$.

We now prove the statement:

> If n has ℓ digits, $\ell \geq 4$, then $\alpha(n)$ has strictly less than ℓ digits.

Suppose ℓ itself has m digits, that is, $10^{m-1} \leq \ell < 10^m$. Then the number of odd digits of n, number of even digits of n, and total number of digits of n, are each at most m. Hence $\alpha(n)$ has at most $3m$ digits.

For $m \geq 2$, we have $3m < 10^{m-1}$, since

$$
\begin{aligned}
10^{m-1} = (1+9)^{m-1} &\geq 1 + 9(m-1), && \text{by Bernouilli's Inequality} \\
&= 1 + 9m - 9 \\
&\geq 1 + 3m + 12 - 9, && \text{since } m \geq 2 \\
&> 3m;
\end{aligned}
$$

so $3m < \ell$ and $\alpha(n)$ has strictly fewer digits than n.

For $m = 1$, we have $3m = 3 < \ell$ since $\ell \geq 4$ and $\alpha(n)$ has strictly fewer digits than n.

In conclusion, if n has three digits or fewer, $\alpha^i(n) = 123$ for i sufficiently large (in fact, $i \leq 5$); if n has more than three digits, successive applications of α give numbers with progressively fewer digits, until a number is reached with three or fewer digits, and then we can apply the previous statement (so that, we at least have $i \leq \log_{10}(n) + 5$).

8. Polynomials

Does there exist a polynomial $p(x, y)$ with real coefficients such that $p(m, n)$ is a non-negative integer (i.e. in $\mathbb{Z}_{\geq 0}$), if $m, n \in \mathbb{Z}_{\geq 0}$ and such that $p : (\mathbb{Z}_{\geq 0})^2 \to \mathbb{Z}_{\geq 0}$ is a bijection?

Note. A polynomial $p(x, y)$, i.e. one in two variables, becomes a polynomial in one variable by setting either variable to a constant. For example, take $p(x, y) = 2x^3 + xy^2 + 7xy + y^2$.

Solution. **[Inspired by Aaron Maynard, 3rd year, Saul Freedman, 1st year, UWA]**
À la "Cantor's zig-zag" argument, we number the pairs of non-negative integers (x, y) in the following way:

		0	1	2	3	4	5	$\cdots$
					x			
	0	0	1	3	6	10	15	
	1	2	4	7	11	16		
	2	5	8	12	17			
y	3	9	13	18				
	4	14	19					
	5	20						
	$\vdots$							

More precisely we define p, recursively, as follows:

$$p(0, 0) = 0,$$
$$p(x, 0) = p(0, x - 1) + 1, \qquad \text{if } x \geq 1,$$
$$p(x, y) = p(x + 1, y - 1) + 1, \text{ if } y \geq 1.$$

Observe that every non-negative integer appears exactly once in the body of the table. Hence p is a bijection as required. We want to show that p is a polynomial with real coefficients. We claim that $p(x, 0) = \frac{1}{2}x(x + 1)$. This is trivially true for $x = 0$. Assume it is true for x, and let us deduce it for $(x + 1)$. We have

$$
\begin{aligned}
p(x + 1, 0) &= p(0, x) + 1 \\
&= p(1, x - 1) + 2 \\
&\ \ \vdots \\
&= p(x, 0) + x + 1 \\
&= \tfrac{1}{2}x(x + 1) + (x + 1) \\
&= \tfrac{1}{2}(x + 1)(x + 2).
\end{aligned}
$$

Hence the claim follows by induction. Now

$$
\begin{aligned}
p(x, y) &= p(x + 1, y - 1) + 1 \\
&= p(x + 2, y - 2) + 2 \\
&\ \ \vdots \\
&= p(x + y, y - y) + y \\
&= p(x + y, 0) + y \\
&= \tfrac{1}{2}(x + y)(x + y + 1) + y \\
&= \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + xy + \tfrac{1}{2}x + \tfrac{3}{2}y,
\end{aligned}
$$

which is a polynomial with real coefficients.

9. A map of a square

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a map such that $f(a) + f(b) + f(c) + f(d) = 0$ whenever a, b, c, d are the 4 vertices of a square.

Is it true that $f(x) = 0$ for all $x \in \mathbb{R}^2$?

Solution. [**By Tuo Li, 2nd year, UWA**]

Choose an arbitrary point $X \in \mathbb{R}^2$.
Construct a square $ABCD$, with X as its centre.
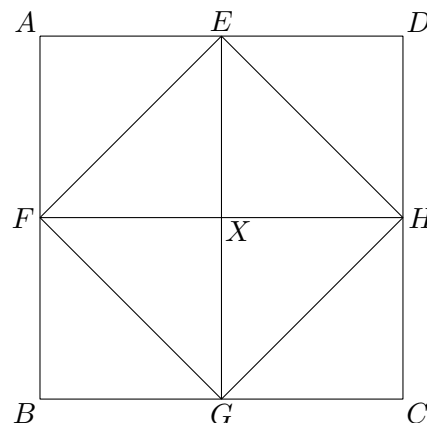Let E, F, G, H be the midpoints of AD, AB, BC, CD, respectively.
Since $AFXE$, $DEXH$, $BGXF$ and $CHXG$ are squares, we have:

$$f(A) + f(F) + f(X) + f(E) = 0 \qquad (7)$$
$$f(D) + f(E) + f(X) + f(H) = 0 \qquad (8)$$
$$f(B) + f(G) + f(X) + f(F) = 0 \qquad (9)$$
$$f(C) + f(H) + f(X) + f(G) = 0 \qquad (10)$$

Adding (7), (8), (9), (10), we have

$$\big(f(A) + f(B) + F(C) + f(D)\big) + 2\big(f(E) + f(F) + F(G) + f(H)\big) + 4f(X) = 0. \qquad (11)$$

But $ABCD$ and $EFGH$ are also squares; hence

$$f(A) + f(B) + f(C) + f(D) = 0 \text{ and}$$
$$f(E) + f(F) + f(G) + f(H) = 0,$$

and so (11) reduces to

$$4f(X) = 0$$
$$\therefore f(X) = 0.$$

Since $x = X$ was arbitrarily chosen, it follows that:
    Yes, it is true that $f(x) = 0$ for all $x \in \mathbb{R}^2$.

10. A prime degree polynomial

A polynomial of degree 2011 with real coefficients is such that $P(n) = \dfrac{n}{n+1}$ for all integers $n \in \{0, 1, 2, \ldots, 2011\}$.

What is the value of $P(2012)$?

Solution. We are given a polynomial $P(x)$ over $\mathbb{R}$ (i.e. with real coefficients) of degree 2011. Let $Q(x) = (x+1)P(x) - x$. Then $Q(x)$ is a polynomial of degree 2012 over $\mathbb{R}$ and for $n \in \{0, 1, 2, \ldots, 2011\}$,

$$Q(n) = (n+1) \cdot \frac{n}{n+1} - n = 0,$$

so that $Q(x)$ has zeros $0, 1, 2, \ldots, 2011$.

Since $Q(x)$ is of degree 2012 and has 2012 distinct zeros, it can have no other zeros. Hence,

$$Q(x) = k \prod_{i=0}^{2011} (x - i)$$

for some real constant k. Now,

$$
\begin{aligned}
1 &= Q(-1) \\
&= k \prod_{i=0}^{2011} (-1 - i) \\
&= k \cdot (-1)^{2012} \cdot \prod_{i=0}^{2011} (i + 1) \\
&= k \cdot 2012! \\
\therefore k &= \frac{1}{2012!} \\
\therefore Q(x) &= \frac{1}{2012!} \prod_{i=0}^{2011} (x - i) \\
\therefore 2013 \cdot P(2012) - 2012 = Q(2012) &= \frac{1}{2012!} \prod_{i=0}^{2011} (2012 - i) \\
&= \frac{1}{2012!} \cdot 2012! = 1 \\
\therefore P(2012) &= \frac{1 + 2012}{2013} \\
&= 1.
\end{aligned}
$$