The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2012 Problems with Solutions

1. A floor in the argument

We denote by $\lfloor r \rfloor$, the *floor* of r, i.e. the greatest integer that is at most the real number r.

Find all the real solutions to the equation,

$$\lfloor x^2 - 3x + 2 \rfloor = 3x - 7.$$

Solution. [**By Dieu Thu Nguyen, 2nd year, Curtin**]
We use the property,

$$\lfloor r \rfloor \le r < \lfloor r \rfloor + 1, \text{ for } r \in \mathbb{R}.$$

Thus,

$$\begin{aligned}
\lfloor x^2 - 3x + 2 \rfloor = 3x - 7 &\iff 3x - 7 \le x^2 - 3x + 2 < (3x - 7) + 1 \text{ and } 3x \in \mathbb{Z} \\
&\iff 0 \le x^2 - 6x + 9 < 1 \text{ and } 3x \in \mathbb{Z} \\
&\iff 0 \le (x-3)^2 < 1 \text{ and } 3x \in \mathbb{Z} \\
&\iff -1 < x - 3 < 1 \text{ and } 3x \in \mathbb{Z} \\
&\iff 2 < x < 4 \text{ and } 3x = n \text{ for some } n \in \mathbb{Z} \\
&\iff 2 < \frac{n}{3} < 4 \text{ and } x = \frac{n}{3}, n \in \mathbb{Z} \\
&\iff 6 < n < 12 \text{ and } x = \frac{n}{3}, n \in \mathbb{Z} \\
&\iff n \in \{7, 8, 9, 10, 11\} \text{ and } x = \frac{n}{3}, n \in \mathbb{Z} \\
&\iff x \in \{\tfrac{7}{3}, \tfrac{8}{3}, 3, \tfrac{10}{3}, \tfrac{11}{3}\}.
\end{aligned}$$

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2. A determination

Let $n \ge 2$. The number of $n \times n$ matrices, whose coefficients are exactly the integers $1, 2 \dots, n^2$, is $(n^2)!$.

What is the sum of the determinants of all these $(n^2)!$ matrices?

Solution. [**By Blake Segler, 3rd year, UWA; Saul Freedman, 2nd year, UWA**]
The sum is zero.
This is so since the $(n^2)!$ matrices form $\frac{1}{2}(n^2)!$ pairs (noting that, since $n \ge 2$, $(n^2)!$ is even), where the members of each pair are such that their determinants are the negatives of one another. Hence the sum of the determinants for each such pair is zero, and hence the sum of all the determinants of all pairs (all $(n^2)!$ matrices) is zero.
To construct the pairing, each matrix is paired with the matrix obtained by interchanging the first two rows (each matrix has at least two rows since $n \ge 2$). The matrix obtained by interchanging the first two rows has the following properties:

(i) It is not the original matrix since the entries in the first two rows are distinct and hence the rows are different.

(ii) It is paired with the original matrix, i.e. if the partner of A is B, then the partner of B is A, since interchanging the first two rows of A, then interchanging those rows again restores A.

(iii) It is one of the $(n^2)!$ matrices, since the interchange of the first two rows operation effects a permutation of the entries $1, 2 \dots, n^2$, and our set of matrices include the matrices with all such $(n^2)!$ permutations.

(iv) It is a matrix with determinant equal to the negative of the determinant of the original matrix (this follows from the standard property that interchanging any two rows of a matrix changes the sign of its determinant, while leaving its magnitude unchanged).

Since we have shown we can construct a pairing of the $(n^2)!$ matrices with the necessary properties, it follows that the sum of the determinants of all the $(n^2)!$ matrices is indeed 0.

Alternatively, let the matrices be A_i, $i = 1, \dots, (n^2)!$. Let τ be the map that swaps the first two rows of a matrix. Then $\tau(A_i)$, $i = 1, \dots, (n^2)!$, are the same matrices, merely permuted, as per the claims above. So

$$\begin{aligned}\sum_i |A_i| &= \sum_i |\tau(A_i)| = \sum_i -|A_i| \\ \therefore 2\sum_i |A_i| &= 0 \\ \sum_i |A_i| &= 0.\end{aligned}$$

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3. Moments with functions

Find all functions $f(x)$, that are continuous and non-negative on $[0, 1]$, such that

$$\int_0^1 f(x)\,dx = 1, \quad \int_0^1 f(x)x\,dx = a \quad \text{and} \quad \int_0^1 f(x)x^2\,dx = a^2,$$

where a is a given real number.

Solution. [**Based on solution of Blake Segler, 3rd year, UWA**]
Suppose a function f with the given properties exists. Then, since f is continuous such that $f(x) \geq 0$ for $x \in [0, 1]$ and $\int_0^1 f(x)\,dx = 1$, f is the pdf of a continuous random variable X on $[0, 1]$. Now, consider, $\int_0^1 f(x)(x-a)^2\,dx$.

$$\begin{aligned}\int_0^1 f(x)(x-a)^2\,dx &= \int_0^1 f(x)(x^2 - 2ax + a^2)\,dx \\ &= \int_0^1 f(x)x^2\,dx - 2a\int_0^1 f(x)x\,dx + a^2\int_0^1 dx \\ &= a^2 - 2a\cdot a + a^2\cdot 1 \\ &= 0.\end{aligned}$$

But $f(x) \geq 0$ for $x \in [0, 1]$ and $(x-a)^2 \geq 0$ (squares of real numbers are non-negative). So, we have

$$f(x)(x-a)^2 \geq 0 \text{ for } x \in [0,1], \text{ and } \int_0^1 f(x)(x-a)^2\,dx = 0.$$

and hence, since $f(x)$ is continuous,

$$f(x)(x-a)^2 = 0 \text{ for } x \in [0,1].$$

Now, $(x-a)^2 = 0 \iff x = a$. Hence,

$$f(x) = 0 \text{ for } a \neq x \in [0,1]$$

and by invoking continuity of f again we deduce that

$$f(x) = 0 \; \forall x \in [0,1]$$

which implies

$$\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0 \text{ (contradiction)}\lightning$$

Thus, there exist no functions f with all the prescribed properties.

Remark. The properties show that $E(X) = a$ and $\mathrm{Var}(X) = 0$:

$$\begin{aligned}
\int_0^1 f(x)x\,dx = a &\implies \mu_X = E(X) = a \text{ (and so } a \in [0,1] \text{, since } X \in [0,1])\\
\int_0^1 f(x)x^2\,dx = a^2 &\implies E(X^2) = a^2\\
&\implies \mathrm{Var}(X) = E(X^2) - E(X)^2 = a^2 - a^2 = 0,
\end{aligned}$$

from which we may deduce that $f(x) = 0$ a.e. (almost everywhere), and hence, with continuity of f we have the deductions made above.

Extension. The properties,

$$f(x) = 0, \text{ for } x = a \in [0,1], \int_0^1 f(x)\,dx = 1$$

say that $f(x)$ has the properties of the Dirac delta "function" $\delta(x-a)$ for some $a \in [0,1]$, which is visualised as being zero except for an infinite height spike at $x = a$, the area under which is 1.

However, $\delta(x-a)$ is not continuous and is not a function, although it can be realised as the limit of a sequence of continuous functions.

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4. Intersecting graphs

For which real numbers $a > 1$ do the graphs of a^x and $\log_a x$ intersect?

Solution. [**By Dieu Thu Nguyen, 2nd year, Curtin**]
Since a^x and $\log_a x$ are inverse functions of each other, their graphs are the reflections of one another in the line $y = x$, and hence any intersection points of the graphs of a^x and $\log_a x$ are along the line $y = x$.

Put another way, the graphs of a^x and $\log_a x$ intersect if and only if the function $f(x) = a^x - x$ has a zero. Now,

$$\begin{aligned}
f'(x) &= a^x \ln a - 1\\
&= 0 \iff a^x \ln a = 1\\
& \iff a^x = (\ln a)^{-1}\\
& \iff x = -\log_a(\ln a) = \log_a\big((\ln a)^{-1}\big)\\
f''(x) &= a^x(\ln a)^2 > 0 \; \forall x \in \mathbb{R}\\
\therefore \min f(x) &= f\big(-\log_a(\ln a)\big)\\
&= (\ln a)^{-1} + \log_a(\ln a)
\end{aligned}$$

Now, $\lim_{x\to\infty} f(x) = \infty$ and $f(x)$ is continuous on $\mathbb{R}$. Hence,

$$\text{Range}\, f(x) = [\min f(x), \infty).$$

Thus,

$$\begin{aligned}
f(x) \text{ has a zero } &\iff \min f(x) \le 0 \\
&\iff (\ln a)^{-1} + \log_a(\ln a) \le 0 \\
&\iff \log_a(\ln a) \le -\frac{1}{\ln a} \\
&\iff \ln a \cdot \log_a(\ln a) \le -1, \text{ since } a > 1 \text{ implies } \ln a > 0 \\
&\iff \ln a \cdot \frac{\ln(\ln a)}{\ln a} \le -1 \\
&\iff \ln(\ln a) \le -1 \\
&\iff \ln a \le e^{-1} \\
&\iff a \le e^{e^{-1}}.
\end{aligned}$$

Thus, for $1 < a \le e^{e^{-1}}$, the graphs of a^x and $\log_a x$ intersect.

Remark. At the last two steps we exponentiated both sides of an inequality. This is legal since $\exp(x)$ is a strictly increasing function for all $x \in \mathbb{R}$, and strictly increasing functions preserve the directionality of an inequality. This follows from the definition:

g is strictly increasing if and only if $x < y \implies f(x) < f(y)$.

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5. Cut the pack

Let $n \ge 1$. A stack of $2n+1$ cards marked 1 or -1 in random order has more 1s than -1s.

Show that it is possible to cut the stack at most once, putting the top part below the bottom part, so that enumerating the cards sequentially from top to bottom, the totals are always positive.

Solution. [**By Sahil Khubchandani, 2nd year, UWA**]

Let T_j be the cumulative total for card j, numbering from the top of the deck. If all the T_j are positive we are done, and hence do not cut.

Otherwise, determine the minimum T_j, $-m$ say (which is non-positive, i.e. $-m \le 0$), and if it occurs more than once take the one for which j is largest, k say. Then by cutting at card k, so that card $k+1$ becomes the new top card, we have removed from in front of that card, cards that total $-m$. Hence the new cumulative totals for cards $k+1$ to $2n+1$ increases by m. Now $T_j > -m$, for $k+1 \le j \le 2n+1$, since the last minimum T_j was at card k, and so the new cumulative totals for cards $k+1$ to $2n+1$ is

$$T_j + m > -m + m = 0,$$

i.e. the new cumulative totals of the cards $k+1$ to $2n+1$ are all positive.

Now T_{2n+1} is the total of all the cards, and we are given that there are more 1s than -1s. So $T_{2n+1} \ge 1$. After the cut, the new cumulative total of card $2n+1$, having increased by m, is

$\geq m+1$, and so the cumulative totals of cards 1 to k, all increase by $\geq m+1$, i.e. the new cumulative totals of cards 1 to k is

$$\geq T_j + m + 1 \geq -m + m + 1 = 1.$$

Thus after the cut all cumulative totals are positive.

So we need cut at most once to ensure all cumulative totals are positive.

Alternative solution (by induction). [**By Dieu Thu Nguyen, 2nd year, Curtin**]
The base case has 3 cards, which since there are more 1s than -1s, must be either three 1s or two 1s and a -1. For three 1s no cut is needed. For two 1s and a -1, we cut below the -1 if necessary, so that we obtain $1, 1, -1$ (from top to bottom) which has all cumulative totals positive.

For the inductive step, we have two cases. Either we cannot find a 1 followed by a -1, in which case all the 1s are together at the bottom of the deck, and all the -1s are at the top; so we cut below all the -1s and so move all the 1s to the top, in which case all cumulative totals are now positive, since there are more 1s than -1s.
Otherwise, there is somewhere a 1 followed by a -1. Remove this pair temporarily, but remember where they came from. By the inductive assumption the remaining cards can be cut at most once so that all cumulative totals are positive. Perform such a cut if one is necessary. Now put the two removed cards back where they would have been. Above the pair all the cumulative totals are unchanged and so positive. The 1 has cumulative total 1 more than the card above it, and so is positive. The -1 has cumulative total the same as the cumulative total of the card above the 1, and so that cumulative total is positive, and all the cards below the -1 have cumulative total the same as with 1 and -1 removed (since $1 + -1 = 0$), and so these cards also have positive cumulative totals.

This completes the induction, and so it follows, that at most one cut is required for any such set of $2n+1$ cards.

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6. A treasure island

A pirate landed on an island in order to hide a treasure. He saw two big rocks A and B, and a bit further inland, 3 palm trees C_1, C_2 and C_3. Starting from C_1, the pirate drew a line segment C_1A_1 perpendicular to and of the same length as C_1A, where A_1 is in a different half-plane with axis AC_1 than B. Then he drew the line segment C_1B_1 perpendicular to and of the same length as C_1B, where B_1 is in a different half-plane with axis BC_1 than A. Then he put a stick at the intersection P_1 of AB_1 and A_1B. Then he did the same procedure with the two palm trees C_2 and C_3. He drew a circle passing through the three sticks P_1, P_2 and P_3, (which were at distinct points) and hid the treasure at the centre of this circle. He then removed the sticks and erased the drawings.

Returning to the island 10 years later to retrieve the treasure, the pirate sees with horror that the island had been hit by a hurricane and that there is no trace of the palm trees anymore. Luckily, the pirate is very clever and managed to retrieve the treasure anyway. How did he do it?

Note. A half-plane with *axis* ℓ, is the entire region on one side of the line ℓ.

Solution. [**By Dieu Thu Nguyen, 2nd year, Curtin**]
We consider the construction of P_1. First we show that

$$\triangle A_1C_1B \cong \triangle AC_1B_1$$

by the SAS Rule. Hence,

$$\angle C_1A_1P_1 = \angle C_1A_1B = \angle C_1AB_1 = \angle C_1AP_1$$

Hence there is a circle that passes through the 4 points A, A_1, C_1, P_1.

$$\begin{aligned} \therefore \angle AP_1A_1 &= \angle AC_1A_1 = 90^\circ \\ \therefore \angle AP_1B &= 180^\circ - \angle AP_1A_1 \\ &= 90^\circ \end{aligned}$$

Hence, P_1 lies on the circle of diameter AB.
Similarly, in constructing P_2 and P_3, we have P_2 and P_3 lie on the circle of diameter AB.
Since P_1, P_2 and P_3 are distinct points (given), there can be at most one circle through P_1, P_2 and P_3, and hence it is the circle with diameter AB, and hence the centre of this circle is the midpoint of AB.
Hence the pirate dug up the treasure at the midpoint of AB.

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7. Indomitable

Is it possible to cover a 6×6 grid with eighteen 2×1 dominoes in such a way that

(i) each of the 5 internal vertical lines forming the grid bisects at least one domino, and
(ii) each of the 5 internal horizontal lines forming the grid bisects at least one domino?

Note. The bisection of a domino by a horizontal or vertical line cuts it into two 1×1 cells.

Solution. [**Based on solution of Blake Segler, 3rd year, UWA**]
Assume we have such a covering of the grid with dominoes. Consider an internal vertical line n, counting from the left. Then to the left of this line there are $6n$ cells. Since by assumption the vertical line cuts a domino, there is a horizontal domino that straddles this line. Suppose for a moment that there is no other horizontal domino that straddles this line. Then there are $6n - 1$ cells to the left of vertical line n, other than the one covered by the horizontal domino straddling vertical line n. But $6n - 1$ is odd and so cannot be covered by 2×1 dominoes. Thus there are at least two horizontal dominoes straddling vertical line n. Since there was nothing special about vertical line n, it follows that all interior vertical lines must be straddled by 2 horizontal dominoes. Since a domino can only straddle one vertical line at at time, these 5×2 horizontal dominoes are distinct.
By symmetry, there are also 2 vertical dominoes straddling each horizontal line, and hence 5×2 distinct vertical dominoes.
But we now have a total of 20 dominoes, but there are only $\frac{36}{2} = 18$ dominoes.
So we have a contradiction, and hence it is not possible to find such a covering.

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8. n-hedra?

For which integers n does there exist in $\mathbb{R}^3$ a polyhedron with exactly n edges?

Solution. [**Based on solutions by Eliot Courtney, 1st year, UWA; Saul Freedman, 2nd year, UWA**]
Let V and E be the numbers of vertices and edges of the polyhedron.
The polyhedron with the fewest edges is a tetrahedron with 6 edges. So $n \geq 6$.
Every vertex is incident with at least 3 edges, and each edge is incident with exactly two vertices; so $2E \geq 3V$ (or $V \leq \frac{2}{3}E$). Also, each pair of vertices is incident with exactly one edge, and (again) each edge is incident with exactly two vertices; so $E \leq \binom{V}{2}$.
Suppose $E = 7$. Then $V \leq \frac{14}{3}$. But $V \in \mathbb{Z}$. So $V \leq 4$, and hence $E \leq \binom{4}{2} = 6$ (contradiction)$\lightning$. So E cannot be 7.
Consider a pyramid with a k-gonal base, where $k \geq 3$. Then the pyramid has $2k$ edges, where k edges make up the base and there is one edge joining each base vertex to the apex of the pyramid. So n can be the even integers $6, 8, 10, \dots$.
Now take one of the base vertices of one of the above pyramids and cut off a tetrahedron, obtaining an extra 3 edges, i.e. $n = 2k + 3$, $k \geq 3$. Hence n may be any odd ≥ 9.
Thus there exist polyhedra with exactly n edges, for exactly $7 \neq n \geq 6$.

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9. Diminishing functions

Do there exist two decreasing functions f, g from $\mathbb{R}$ to $\mathbb{R}$ such that

(a) $f\big(f(x)\big) = x + 1$ for all $x \in \mathbb{R}$?

(b) $g\big(g(x)\big) = 2x + 1$ for all $x \in \mathbb{R}$?

Solution.

(a) [**By Eliot Courtney, 1st year, UWA**]
Suppose such a f exists. Then substitute $f(x)$ in (a):

$$\begin{aligned}
&\implies f\Big(f\big(f(x)\big)\Big) = f(x) + 1\\
&\implies f(x+1) = f(x) + 1, \quad \text{using (a) on the left}\\
&\implies f(x+1) > f(x)\\
&\implies f \text{ is not decreasing (contradiction)}\lightning
\end{aligned}$$

Hence no such f exists.

(b) [**By Sahil Khubchandani, 2nd year, UWA**]
Let $g(x) = -\sqrt{2}x - \dfrac{1}{\sqrt{2}-1} = -\sqrt{2}x - (\sqrt{2}+1)$. Then $g(x)$ is decreasing since it is of the form $ax + b$, with $a = -\sqrt{2} < 0$, and

$$\begin{aligned}
g\big(g(x)\big) &= -\sqrt{2}\Big(-\sqrt{2}x - \frac{1}{\sqrt{2}-1}\Big) - \frac{1}{\sqrt{2}-1}\\
&= 2x + \frac{\sqrt{2}-1}{\sqrt{2}-1}\\
&= 2x + 1.
\end{aligned}$$

We observe that g and $g \circ g$ are defined for all $x \in \mathbb{R}$.
Thus there does exist a function $g : \mathbb{R} \to \mathbb{R}$ such that $g\big(g(x)\big) = 2x + 1 \ \forall x \in \mathbb{R}$.

Remark. The example above is discovered by guessing that there might be an affine solution, i.e. one of form $g(x) = ax + b$ with $a < 0$ to ensure it is decreasing. Substitution in the relation (b), we find $a^2 = 2$ and $b(a + 1)$, and with $a < 0$ we obtain the solution above. But are any solutions necessarily of this form? The answer is: "No", as the following solution demonstrates.

Alternative example. [**By Dieu Thu Nguyen, 2nd year, Curtin**]
Define g to be piecewise continuous, by

$$g(x) = \begin{cases} -2x - 3, & x < -1 \\ -1, & x = -1 \\ -x - 2, & x > -1. \end{cases}$$

Then careful checking shows that $g\big(g(x)\big) = 2x + 1$ and the negative slope of the "pieces" show that g is decreasing.

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10. Food fight

There are $2n + 1$ people standing in a flat field, in such a way that their pairwise distances are distinct. Everybody launches a pie towards their closest neighbour. Show the following.

(a) No pie trajectories intersect (except for cases where two people throw pies at each other).

(b) The network formed by all the pie trajectories has no closed polygon.

(c) At least one person is not hit by any pie.

(d) Nobody is hit by more than 5 pies.

Note. This is a two-dimensional problem. Each pie trajectory is to be thought of as a line segment in the plane of the flat field.

Solution.

(a) [**By Eliot Courtney, 1st year, UWA**]
Suppose for a contradiction that 2 pie trajectories $A \to B$ and $C \to D$ intersect at M. Then since the people launch at their nearest neighbour, we have

$$\begin{aligned} &AB < CD,\ AB < AD \\ &CD < AC,\ CD < BC \\ \therefore\ &AD + BC > AB + CD \\ &\qquad\qquad = AM + MB + CM + MD \\ &\qquad\qquad = (AM + MD) + (CM + MB) \\ &\qquad\qquad > AD + BC \text{ (contradiction)} \lightning, \text{ by the Triangle Inequality} \end{aligned}$$

Therefore, no pie trajectories intersect.

(b) [**By Dieu Thu Nguyen, 2nd year, Curtin**]
Supposefor a contradiction there is a closed polygon such that

$$A_1 \to A_2 \to \cdots \to A_n \to A_1$$

where $\rightarrow$ indicates the direction a pie is launched, and let A_{n+1} be another name for A_1 etc.
Then $A_kA_{k+1} > A_{k+1}A_{k+2}$ for $k \geq 1$ since otherwise A_{k+1} would not launch their pie at A_{k+2}. Hence

$$A_1A_2 > A_2A_3 > \cdots > A_{n-1}A_n > A_nA_1 > A_1A_2$$

with no equalities since all distances are distinct.
Hence $A_1A_2 > A_1A_2$, which is impossible.
Therefore, there is no closed polygon of trajectories.

(c) [**Based on Dieu Thu Nguyen, 2nd year, Curtin**]
Some of the people may throw their pies at each other. This leaves an odd number of people that fall under the restriction of (b).
Suppose for a contradiction that these remaining people are each hit by a pie. Take one of these people B_1. Suppose he is hit by B_2, then B_2 is hit by B_3, and so on. The $B_1, B_2, \ldots$ are distinct by (b). Continuing in this way we eventually exhaust the remaining people, since otherwise we would have someone not hit by a pie. But then the last person admitted to the chain cannot have been by a pie because otherwise someone has thrown two pies or we have a contradiction via (b).
Thus at least one person was not hit by a pie.

(d) [**Based on Dieu Thu Nguyen, 2nd year, Curtin**]
We use the fact that the lengths of sides of a triangle are in the same order as the sizes of the angles they are opposite, e.g. the largest angle is opposite the largest side.
Suppose for a contradiction that A is hit by pies from $B_1, \ldots, B_6$. Let $B_1, \ldots, B_6$ be numbered in order around A. Then

$$\angle B_1AB_2 + \angle B_2AB_3 + \cdots + \angle B_6AB_1 = 360^\circ.$$

Then one of these angles $\leq 60^\circ$ (since if not their sum $> 360^\circ$). Without loss of generality, suppose $\angle B_1AB_2 \leq 60^\circ$.
Consider the triangle B_1AB_2. For brevity call its angles $\angle B_1, \angle A = \angle B_1AB_2, \angle B_2$. Since B_1 and B_2 launch at A, we have

$$\begin{aligned}
B_1A < B_1B_2 &\implies \angle B_2 < \angle A \\
B_2A < B_1B_2 &\implies \angle B_1 < \angle A \\
&\implies \angle B_1 + \angle B_2 + \angle A < 3\angle A \\
&\qquad\qquad\qquad\qquad\quad \leq 3 \cdot 60^\circ = 180^\circ
\end{aligned}$$

i.e. the angle sum of triangle B_1AB_2 is $< 180^\circ$ which is impossible, and we have our contradiction.
Thus no one is hit by more than 5 pies.