The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2014 Problems

1. Polyhistoric accuracy

Does there exist a polynomial $p(x)$ with integer coefficients such that $p(2014) = 1915$ and $p(1788) = 1901$?

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2. Diagonal revolution

Consider the solid of revolution obtained by rotating a rectangle with side lengths ℓ and b, $\ell > b$, along one of its diagonals.

What is the volume of this solid?

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3. Fractionally sequential

Let O_n (resp. E_n) be the number obtained by writing (in base 10) the first n odd (resp. even) numbers one after the other.

Does the sequence of rational numbers O_n/E_n have a limit as n goes to infinity? (The first few terms of the sequence are: 1/2, 13/24, 135/246, 1357/2468, $13579/246810, \ldots$)

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4. Imperfect square

Show that no number of the form $(2n-1)(6n-1)$, where n is a positive integer, can be a perfect square.

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5. Trigonometrically rational

What are the real solutions of the equation

$$\sin x \cdot \cos^2 x \cdot \tan^3 x \cdot \cot^4 x \cdot \sec^5 x \cdot \operatorname{cosec}^6 x = \frac{256}{27}?$$

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6. Powerfully whole

For what positive integers n is $(2013 + \frac{1}{2})^n + (2014 + \frac{1}{2})^n$ also an integer?

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7. Piling matches

Starting with a finite number of matches divided into a finite number of piles, at each step, we perform the following procedure:

> Choose two piles and remove from the bigger one as many matches as there are in the smaller pile and add these to the smaller pile. In the case that the two chosen piles have the same size, they are merged into one, reducing the overall number of piles by one.

For example, if we start with three piles with 3, 12, and 33 matches, respectively, we can do:

$$[3, 12, 33] \to [6, 12, 30] \to [12, 12, 24] \to [24, 24] \to [48]$$

so that we get one pile after 4 steps.

For which of the following five starting situations, is it possible to obtain only one pile after a finite number of steps?

(i) $[1, 2, 3]$

(ii) $[1, 2, 2, 3, 3, 4, 5, 12, 17, 15]$

(iii) $[16, 4, 5, 5, 8]$

(iv) $[51, 54, 75, 75, 81, 48]$

(v) $[1, 2, 3, 4, 5, 6, 7]$

(vi) $[1, 1, 2, 2, 2, 3, 3, 3, 3, 5]$

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8. Tangentially circumcircled

The tangents at B and C to the circumcircle of acute-angled triangle ABC intersect in X. Let M be the midpoint of BC.

Prove that $AM/AX = \cos \angle CAB$.

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9. Tetrahedral triangle

Prove that every tetrahedron has at least one vertex such that the three edges incident with this vertex have lengths equal to the sides of some triangle.

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10. Recursively limited?

Let the sequence $a_1, a_2, \ldots$ be defined by

$$a_1 = \sqrt{2},$$
$$a_{n+1} = (\sqrt{2})^{a_n}, \text{ for } n \geq 1.$$

Does this sequence converge? If so, what is its limit?
