The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2015 Problems

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers).

1. Absolution

What are the solutions $x \in \mathbb{R}$ of the equation

$$|x+1| - |x| + 3|x-1| - 2|x-2| = x + 2?$$

Solution.
Since, for constant $a \in \mathbb{R}$ we may write

$$|x-a| = \begin{cases} x-a, & \text{if } x \geq a \\ a-x, & \text{if } x < a, \end{cases}$$

by consideration over each of the intervals $(-\infty, -1)$, $[-1, 0)$, $[0, 1)$, $[1, 2)$, $[2, \infty)$, we may rewrite

$$|x+1| - |x| + 3|x-1| - 2|x-2| = x + 2 \qquad (*)$$

without absolute values and thus easily solve on each interval.

- On $(-\infty, -1)$. Here $(*)$ becomes

$$\begin{aligned} -1 - x - (-x) + 3(1-x) - 2(2-x) &= x + 2 \\ -4 &= 2x \\ x &= -2. \end{aligned}$$

  Since $-2 \in (-\infty, -1)$, $x = -2$ is a solution of $(*)$.

- On $[-1, 0)$. Here $(*)$ becomes

$$\begin{aligned} (x+1) - (-x) + 3(1-x) - 2(2-x) &= x + 2 \\ x &= x + 2 \end{aligned}$$

  which can never be satisfied. So $(*)$ has no solutions in $[-1, 0)$.

- On $[0, 1)$. Here $(*)$ becomes

$$\begin{aligned} (x+1) - x + 3(1-x) - 2(2-x) &= x + 2 \\ -2 &= 2x \\ x &= -1. \end{aligned}$$

  But $-1 \notin [0, 1)$; so we have a contradiction, and hence $(*)$ has no solutions in $[0, 1)$.

- On $[1, 2)$. Here $(*)$ becomes

$$\begin{aligned} (x+1) - x + 3(x-1) - 2(2-x) &= x + 2 \\ 8 &= 4x \\ x &= 2. \end{aligned}$$

  But $2 \notin [1, 2)$; so again we have a contradiction, and hence $(*)$ has no solutions in $[1, 2)$.

- On $[2, \infty)$. Here $(*)$ becomes

$$\begin{aligned} (x+1) - x + 3(x-1) - 2(x-2) &= x + 2 \\ 0 &= 0, \end{aligned}$$

  which is always true. So every $x \in [2, \infty)$ is a solution of $(*)$.

Taking the union of the solutions found on the separate intervals, we have that the complete set of solutions to $(*)$ is

$$\{-2\} \cup [2, \infty).$$

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2. Non-repetitive integers

How many positive integers have no repeated digits in their decimal representation?

Solution. [**Inspired by Euclid Kosasih, 1st year, and Conway Li, 2nd year, UWA**]
Consider decimal strings of length $k \in \mathbb{N}$.
If $k \leq 10$ then the number of decimal strings of length k with no repeated digits is

$$^{10}\mathrm{P}_k = 10 \cdot 9 \cdots (10 - k + 1) = \frac{10!}{(10-k)!}$$

of which

$$^{9}\mathrm{P}_{k-1} = 1 \cdot 9 \cdots (10 - k + 1) = \frac{9!}{(10-k)!}$$

have a leading 0. Positive integers are decimal strings without a leading zero. Hence the number of k-digit numbers with no repeated digits is

$$\frac{10!}{(10-k)!} - \frac{9!}{(10-k)!} = \frac{9 \cdot 9!}{(10-k)!}.$$

If $k > 10$ then by the Pigeon Hole Principle, at least one digit will be repeated. Hence the total number of positive integers with no repeated digits is

$$\begin{aligned}
\sum_{k=1}^{10} \frac{9 \cdot 9!}{(10-k)!} &= 9 \cdot (1 + 9 + 9 \cdot 8 + \cdots + 9! + 9!) \\
&= 9 \cdot (1 + 9(1 + 8(1 + 7(1 + 6(1 + 5(1 + 4(1 + 3(1 + 2(1 + 1)) \\
&= 9 \cdot (1 + 9(1 + 8(1 + 7(1 + 6(1 + 5(1 + 4(1 + 3(1 + 4)) \\
&= 9 \cdot (1 + 9(1 + 8(1 + 7(1 + 6(1 + 5(1 + 4(1 + 15)) \\
&= 9 \cdot (1 + 9(1 + 8(1 + 7(1 + 6(1 + 5(1 + 64)) \\
&= 9 \cdot (1 + 9(1 + 8(1 + 7(1 + 6(1 + 325)) \\
&= 9 \cdot (1 + 9(1 + 8(1 + 7(1 + 1956)) \\
&= 9 \cdot (1 + 9(1 + 8(1 + 13\,699)) \\
&= 9 \cdot (1 + 9(1 + 109\,600)) \\
&= 9 \cdot (1 + 986\,409) \\
&= 8\,877\,690
\end{aligned}$$

The factorials suggest that we might be able to express the number in terms of e. Let N_{10} be the number of positive integers with no repeated digits in their decimal representation. Then above we have shown

$$N_{10} = \sum_{k=1}^{10} \frac{9 \cdot 9!}{(10-k)!} = 9 \sum_{k=1}^{10} {}^{9}\mathrm{P}_{k-1} = 9 \sum_{k=0}^{9} {}^{9}\mathrm{P}_k\,.$$

For $n > 1$, let

$$\begin{aligned} S(n) &= \sum_{k=0}^{n} {}^{n}\mathrm{P}_k \\ &= n!\Big(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!}\Big), && \text{writing it backwards} \\ &= n!\Big(e - \Big(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\Big)\Big) \\ &= n!e - \Big(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots\Big) \\ &= n!e - g(n) \end{aligned}$$

where

$$\begin{aligned} g(n) &= \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots \\ &< \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \ldots \\ &= \frac{1/n}{1-(1/n)}, && \text{since it is the sum to infinity of a geometric series where the common ratio has magnitude } 1/n < 1. \\ &= \frac{1}{n-1}. \end{aligned}$$

If $n > 2$ then $g(n) < 1$ and in this case we have

$$n!e - 1 < n!e - g(n) = S(n) < n!e$$

and since $S(n) \in \mathbb{Z}$, because it is a sum of integers of form ${}^{n}P_k$,

$$S(n) = \lfloor n!e \rfloor,$$

and hence

$$N_{10} = 9S(9) = 9\lfloor 9!e \rfloor.$$

Generalisation. We now consider the more general problem of finding the number N_b of integers in base b with no repeating digits. For $b = 2$, the non-repetitive integers are 1_{binary} and 10_{binary}, so that

$$N_2 = 2.$$

For $b > 2$, the arguments above still hold with b in place of 10 (ten), so that

$$\begin{aligned} N_b &= (b-1)\sum_{k=1}^{b} {}^{b-1}\mathrm{P}_{k-1} \\ &= (b-1)\sum_{k=0}^{b-1} {}^{b-1}\mathrm{P}_k \\ &= (b-1)S(b-1) \\ &= (b-1)\lfloor (b-1)!e \rfloor. \end{aligned}$$

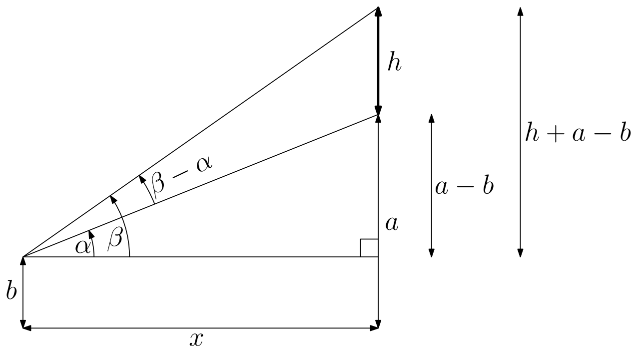
Observe that N_2 also happens to satisfy the above formula. So, in fact

$$N_b = (b-1)\lfloor (b-1)!e \rfloor, \ \forall\, b \geq 2, b \in \mathbb{N}.$$

3. Optically optimal

A picture h metres high is hung on a high wall such that the bottom of the picture is a metres above the ground. A person stands on the ground at a distance x metres from the foot of the wall, viewing the picture. If the person's eye level is b metres above the ground, assuming $a \geq b$, find an expression for the viewing angle of the picture in terms of x, a, b and h. Hence find, for fixed a, b, h, the distance x from the foot of the wall that a person should stand such that the viewing angle is largest.

Solution. [**Elegant solution by Alexander Chua, 2nd Year, UWA.**]
As shown in the diagram, the viewing angle of the picture is $\beta - \alpha$, where necessarily $0 < \beta - \alpha < \pi/2$. We have



$$\tan\alpha = \frac{a-b}{x} \quad \text{and} \quad \tan\beta = \frac{h+a-b}{x}.$$

Hence,

$$\begin{aligned}\tan(\beta-\alpha) &= \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha}\\ &= \frac{h}{x + \dfrac{(h+a-b)(a-b)}{x}}.\end{aligned}$$

Now, for two positive numbers A and B the inequality of arithmetic and geometric means states that

$$\frac{A+B}{2} \geq \sqrt{AB} \implies \frac{1}{A+B} \leq \frac{1}{2\sqrt{AB}},$$

with equality iff $A = B$. So we have

$$\tan(\beta-\alpha) = \frac{h}{x + \dfrac{(h+a-b)(a-b)}{x}} \leq \frac{h}{2\sqrt{(h+a-b)(a-b)}},$$

with equality iff

$$\begin{aligned}x &= \frac{(h+a-b)(a-b)}{x}\\ \iff x &= \sqrt{(h+a-b)(a-b)} \qquad \text{since } x > 0.\end{aligned}$$

Since the angle $\beta - \alpha$ is acute, maximising $\beta - \alpha$ is equivalent to maximising $\tan(\beta - \alpha)$, hence the viewing angle is maximised when when the person stands $\sqrt{(h+a-b)(a-b)}$ metres from the foot of the wall.

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4. Limiting sine times

Evaluate $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{\alpha}{3^n}\right)\sin\left(\frac{2\alpha}{3^n}\right)$.

Solution.
Using the identity,

$$\sin x \sin y = \tfrac{1}{2}\big(\cos(x-y) - \cos(x+y)\big),$$

we have a telescopic sum,

$$\begin{aligned}\sum_{n=1}^{N} \sin\Big(\frac{\alpha}{3^n}\Big)\, \sin\Big(\frac{2\alpha}{3^n}\Big) &= \sum_{n=1}^{N} \tfrac{1}{2}\bigg(\cos\Big(\frac{\alpha}{3^n}\Big) - \cos\Big(\frac{\alpha}{3^{n-1}}\Big)\bigg)\\ &= \tfrac{1}{2}\bigg(\cos\Big(\frac{\alpha}{3^N}\Big) - \cos\alpha\bigg)\\ \therefore\ \sum_{n=1}^{\infty} \sin\Big(\frac{\alpha}{3^n}\Big)\, \sin\Big(\frac{2\alpha}{3^n}\Big) &= \lim_{N\to\infty} \tfrac{1}{2}\bigg(\cos\Big(\frac{\alpha}{3^N}\Big) - \cos\alpha\bigg)\\ &= \tfrac{1}{2}(1-\cos\alpha)\\ &= \sin^2(\tfrac{1}{2}\alpha).\end{aligned}$$

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5. Zeroing in on an integer

Let $p(x)$ be a polynomial with integer coefficients such that $p(2) = 13$ and $p(10) = 5$.

If there exists an integer n such that $p(n) = 0$, what is n?

Solution. [**Elegant solution by Alexander Chua, 2nd Year, UWA.**]

First we prove the following lemma, noting that we say, for $a, b \in \mathbb{Z}$ with $a \neq 0$, that "a *divides* b" (and write: $a \mid b$), if $a = qb$ for some $q \in \mathbb{Z}$.

Lemma. *Let $q(x)$ be a polynomial with integer coefficients. Then $(a-b) \mid \big(q(a) - q(b)\big)$ for all $a, b \in \mathbb{Z}$ where $a \neq b$.*

Proof. Let $q(x) = c_m x^m + c_{m-1}x^{m-1} + \cdots + c_0$, where $c_k \in \mathbb{Z}$ for $0 \leq k \leq m$, and let $a \neq b$. Then

$$\begin{aligned}q(a) - q(b) &= (c_m a^m + c_{m-1}a^{m-1} + \cdots + c_0) - (c_m b^m + c_{m-1}b^{m-1} + \cdots + c_0)\\ &= c_m(a^m - b^m) + c_{m-1}(a^{m-1} - b^{m-1}) + \cdots + c_1(a-b). \qquad (1)\end{aligned}$$

But, in general, we have the factorisation

$$a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \cdots + b^{k-1}),$$

and hence $a - b$ divides each term of (1), and so

$$(a-b) \mid \big(q(a) - q(b)\big),$$

as required. □

Now let's return to the problem. Assume $n \in \mathbb{Z}$ such that $p(n) = 0$. Then $n \neq 2$, since $p(n) = 0 \neq 13 = p(2)$. Similarly, $n \neq 10$. So we can apply the lemma:

$$\begin{aligned}p(n) = 0,\ p(2) = 13 \implies (n-2) \mid 13 &\implies n - 2 = 1, -1, 13 \text{ or } -13\\ &\implies n = 3, 1, 15 \text{ or } -11 \qquad (2)\\ p(n) = 0,\ p(10) = 5 \implies (n-10) \mid 5 &\implies n - 10 = 1, -1, 5 \text{ or } -5\\ &\implies n = 11, 10, 15 \text{ or } 5 \qquad (3)\end{aligned}$$

From (2) and (3), we see that the only value possible for n is 15.

Remark: the polynomial $p(x) = 15 - x$ satisfies $p(2) = 13$, $p(10) = 5$ and $p(15) = 0$. So, for that polynomial (but there could be others), there does indeed exist an unique $n \in \mathbb{Z}$ such that $p(n) = 0$, namely $n = 15$.

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6. Triangular circulation

Four equal circles are drawn inside an equilateral triangle of side length 2 units. Each circle touches two of the other circles and only one side of the triangle.

Find the exact common radius of the circles.

Solution. [**Elegant solution by Alexander Chua, 2nd Year, UWA.**]
Let r be the common radius of the circles. Construct the lines shown in the diagram. Then

$$BE = BD - ED = 1 - r \tag{4}$$

$$\frac{YZ}{BZ} = \tan(60^\circ) = \sqrt{3} \tag{5}$$

$$\therefore BZ = \frac{YZ}{\sqrt{3}}$$

$$= \frac{3r}{\sqrt{3}}, \qquad \text{since } YZ = FE = 3r$$

$$= \sqrt{3}r \tag{6}$$

$$\angle XYF = 180^\circ - \angle FYX - \angle BYZ, \qquad \text{since } \angle BYX \text{ is a straight angle}$$

$$= 180^\circ - 90^\circ - \angle BYZ$$

$$= 90^\circ - 30^\circ, \qquad \text{since } \triangle BYZ \sim \triangle BAD \text{ is a half equilateral triangle}$$

$$= 60^\circ$$

$$\therefore \frac{XF}{YF} = \sin(60^\circ) = \frac{\sqrt{3}}{2}$$

$$\therefore ZE = YF = \frac{2XF}{\sqrt{3}} = \frac{2r}{\sqrt{3}} \tag{7}$$

$$\therefore 1 - r = BE, \qquad \text{by (4)}$$

$$= BZ + ZE$$

$$= \sqrt{3}r + \frac{2r}{\sqrt{3}}, \qquad \text{by (6) and (7)}$$

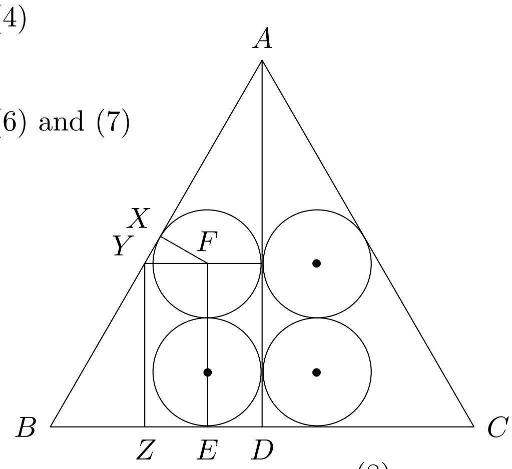
$$\therefore \sqrt{3} - \sqrt{3}r = 3r + 2r$$

$$\sqrt{3} = (5 + \sqrt{3})r$$

$$\therefore r = \frac{\sqrt{3}}{5 + \sqrt{3}}$$

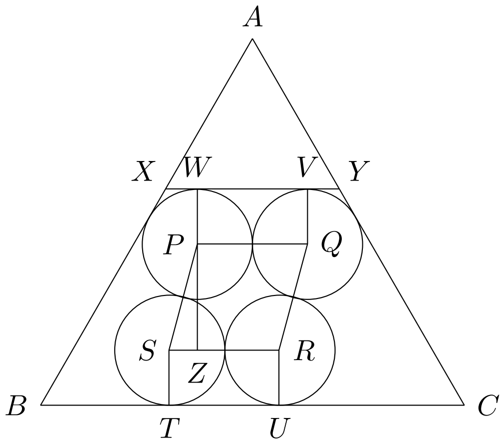
$$= \frac{\sqrt{3}(5 - \sqrt{3})}{(5 + \sqrt{3})(5 - \sqrt{3})}$$

$$= \frac{5\sqrt{3} - 3}{22} \tag{8}$$



So the common radius of the circles is $(5\sqrt{3} - 3)/22$.
More generally, the centres of the circles form a rhombus.

Again, let r be the common radius of the circles, and construct the lines shown in the diagram and let α be the acute angle of the rhombus $PQRS$.
Note that $PWVQ$ and $TSRU$ are rectangles, and $PQRS$ is a rhombus.
So $WV \parallel PQ \parallel RS \parallel TU$. Hence $XY \parallel BC$, and so $\triangle AXY$ is equilateral.
PX bisects $\angle BXY = 120°$.



$$\begin{aligned}
\therefore \angle PXW &= 60° \\
\therefore XW &= \frac{PW}{\tan(60°)} = \frac{r}{\sqrt{3}} \\
\text{Similarly, } YV &= \frac{r}{\sqrt{3}} \\
\therefore AX = XY &= XW + WV + VY = \frac{2r}{\sqrt{3}} + 2r \\
\text{distance}(X, BC) &= WP + PZ + ST \\
&= r + 2r\sin\alpha + r \\
&= 2r(1 + \sin\alpha) \\
\therefore BX &= \frac{\text{distance}(X, BC)}{\cos(30°)} = \frac{4r(1+\sin\alpha)}{\sqrt{3}} \\
\therefore 2 &= AB \\
&= AX + BX \\
&= \frac{2r}{\sqrt{3}} + 2r + \frac{4r(1+\sin\alpha)}{\sqrt{3}} \\
&= \frac{2r + 2\sqrt{3}r + 4r(1+\sin\alpha)}{\sqrt{3}} \\
&= \frac{r(6 + 2\sqrt{3} + 4\sin\alpha)}{\sqrt{3}} \\
\therefore r &= \frac{\sqrt{3}}{3 + \sqrt{3} + 2\sin\alpha}.
\end{aligned}$$

If we put $\alpha = 90°$, we rediscover the value for r when $PQRS$ is a square.

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7. Integer surprise

An integer $n > 0$ is said to be *surprising* if, when written (in the decimal system) to the right of any positive integer, the resulting number is divisible by n.

What are all surprising integers?

Solution. [**Elegant solution by Alexander Chua, 2nd year, UWA**]
Let n be a surprising number with m digits, say, and let $q \in \mathbb{N}$.
Then n written to the right of q is the decimal number $q \cdot 10^m + n$. Now

$$\begin{aligned}
n \,|\, (q10^m + n)\ \forall q \in \mathbb{N} &\iff n \,|\, (10^m + n) \quad \text{(the case with } q = 1) \\
&\iff n \,|\, 10^m.
\end{aligned}$$

Since n has m digits, we have $10^{m-1} \le n < 10^m$.
Hence $1 < 10^m/n \le 10$. However, $10^m/n$ is a positive integer whose only prime factors are

2 and 5, so $10^m/n = 2, 4, 5, 8, 10$. Thus $n = 5 \cdot 10^{m-1}, 25 \cdot 10^{m-2}, 2 \cdot 10^{m-1}, 125 \cdot 10^{m-3}$ or 10^{m-1}, respectively.
This argument not only shows that if n is a surprising number, then it has the above form (where the integer m satisfies $m \geq 1$, $m \geq 2$, $m \geq 1$, $m \geq 3$, $m \geq 1$, respectively), but also shows that numbers of this form are surprising.

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8. Battleship 1-D

A ship, represented by a point, moves in uniform motion along the real line $\mathbb{R}$. At any moment, the ship's position and speed are not known. The only information available is the following:

(i) its position at time $t = 0$ is an integer x.

(ii) the speed (measured per minute) is an integer v.

Every minute from $t = 0$, we drop a bomb on a lattice point (i.e. a point with coordinate $n \in \mathbb{Z}$). If the ship is there, it sinks and we have won.

Is there a strategy guaranteeing the sinking of the ship in finite time?

Solution.
There is a strategy to sink the ship in finite time. Indeed, the set of all possible initial conditions of the ship, i.e. all pairs (*initial-position*, *speed*), is $\mathbb{Z}^2$, the set of all pairs of integers, which is countable, which is to say that the ordered pairs of $\mathbb{Z}^2$ can be put into one-to-one correspondence with the natural numbers $\mathbb{N}$. To see this, for example, start at the origin $(0, 0)$ and track an infinite polygonal line spiralling around the origin, to obtain a sequence of pairs as follows:

$(0, 0), (0, 1), (1, 1), (1, 0), (1, -1), (0, -1), (-1, -1), (-1, 0), (-1, 1), (-1, 2), (0, 2), \ldots, (x_n, v_n), \ldots$

At any time $t = n$, the n^{th} pair of this sequence is chosen and a bomb is dropped at the lattice point $x = x_n + nv_n$. With this strategy, if the ship left point $x_n \in \mathbb{Z}$ with velocity $v_n \in \mathbb{Z}$ then the ship will be sunk at time $t = n$. Thus, since there is a finite n for each possible initial condition pair, the ship is guaranteed to be sunk with the above strategy in finite time n.
In fact, the above strategy works even if the initial position and speed are rational, since $\mathbb{Q}^2$ is still a countable set.

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9. Parity game

Alice chooses 2000 distinct numbers from the set of integers from 1 to 3000. Ben then tries to find among these 2000 numbers, 1000 integers whose parity alternates when they are ordered from smallest to largest. If Ben is able to achieve his objective, he wins; otherwise, Alice wins.

Assuming they each use an optimal strategy, who of Alice or Ben is assured of winning, whatever their opponent does?

Solution.
Ben is assured of victory regardless of which 2000 numbers are selected by Alice. Indeed, consider the 1500 pairs $(1, 2), (3, 4), (5, 6), \ldots, (2k-1, 2k), \ldots, (2999, 3000)$. Eliminate from this set any pair for which at least one partner was not chosen by Alice. At most 1000 pairs will have been eliminated, leaving at least 500 pairs where both members of the pair were selected by Alice. By choosing 500 of these pairs of numbers, in the order they were originally, Ben has 1000 integers ordered from smallest to largest, whose parity alternates.

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10. Multinomial integers

What are the solutions of the equation

$$x_1^{2015} + 2^1 x_2^{2015} + 2^2 x_3^{2015} + \cdots + 2^{2014} x_{2015}^{2015} = 2014 x_1 x_2 x_3 \cdots x_{2015}$$

for $x_1, x_2, x_3, \ldots, x_{2015} \in \mathbb{Z}$?

Solution.
The only solution of the given equation, which henceforth we will refer to as $(*)$, is that all of the integers x_i are zero. Indeed, since from the second member of $(*)$ on, all terms are even, and the right hand side of $(*)$ is even, we deduce that x_1 is even. If we put $x_1 = 2y_1$, with $y_1 \in \mathbb{Z}$, and divide through by 2, the equation $(*)$ becomes

$$x_2^{2015} + 2^2 x_3^{2015} + \cdots + 2^{2013} x_{2015}^{2015} + 2^{2014} y_1^{2015} = 2014 x_2 x_3 \cdots x_{2015} y_1$$

which is $(*)$ again, but with variables $x_2, x_3, x_4, \ldots, x_{2015}, y_1$ instead of $x_1, x_2, x_3, x_4, \ldots, x_{2015}$. This implies that if $(x_1, x_2, \ldots, x_{2015})$ is a solution of $(*)$, then so is $(x_2, x_3, \ldots, x_{2015}, x_1/2)$. Similarly, $(x_3, x_4, \ldots, x_{2015}, x_1/2, x_2/2)$ is a solution, and, iterating this process, eventually we obtain that $(x_1/2, x_2/2, \ldots, x_{2015}/2)$ is an integer solution. Therefore, all the integers in a solution of $(*)$ are necessarily even.
Therefore, if x_i were not all zero, by repeating the above reasoning, eventually the process above would lead to a a solution containing at least one odd x_i, contradicting what we have proved above.
