The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2016 Problems

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers).

1. Digitally prodigious

A number N is formed by concatenating the digits of the numbers $2, 2^2, \dots, 2^{2016}$ and the numbers $5, 5^2, \dots, 5^{2016}$ in their usual decimal representation.
How many digits does N have?
If the second sequence of numbers were powers of 8, instead of 5, and each of the numbers were written as hexadecimals, how many digits would N have then?

Solution. Let $d(n)$ be the number of digits of a natural number n in its decimal representation. Suppose an integer n has k digits. Then

$$\begin{aligned} 10^{k-1} &\le n < 10^k \\ \therefore k-1 &\le \log_{10} n < k \\ \therefore d(n) &= \lfloor \log_{10} n \rfloor + 1. \end{aligned}$$

Observe that

$$\log_{10}(2^m) + \log_{10}(5^m) = \log_{10}(10^m) = m.$$

Suppose $\lfloor \log_{10}(2^m) \rfloor = k - 1$. Then, since $\log_{10}(2^m)$ and $\log_{10}(5^m)$ are irrational,

$$\begin{aligned} k-1 &< \log_{10}(2^m) && < k \\ \therefore -k &< -\log_{10}(2^m) && < -k+1 \\ \therefore m-k &< m - \log_{10}(2^m) \\ &= \log_{10}(5^m) && < m-k+1. \end{aligned}$$

Therefore,

$$\begin{aligned} d(2^m) + d(5^m) &= (k-1) + 1 + (m-k) + 1 \\ &= m + 1 \\ \therefore d(N) &= \sum_{m=1}^{2016} d(2^m) + \sum_{m=1}^{2016} d(5^m) \\ &= \sum_{m=1}^{2016} (m+1) \\ &= 2 + 3 + \cdots + 2017 \\ &= \frac{2016(2+2017)}{2} \\ &= 2\,035\,152. \end{aligned}$$

Hence, N has $2\,035\,152$ digits.

Is there a difference if the second sequence of numbers are powers of 8 instead of powers of 5, and the numbers are now written as hexadecimals? Indeed, there is a difference, essentially due to $\log_{16}(2^m)$ and $\log_{16}(8^m)$ being rational. Let $\delta(n)$ be the number of digits of a natural number n when written as a hexadecimal. Then, analogously, we have

$$\delta(n) = \lfloor \log_{16} n \rfloor + 1.$$

Suppose $\log_{16}(2^m) = k \in \mathbb{Z}$, which is the case whenever m is a multiple of 4. Then $\log_{16}(8^m) = m - k$, and so, in this case,

$$\delta(2^m) + \delta(8^m) = k + 1 + (m - k) + 1 = m + 2.$$

So we get an extra digit whenever m is a multiple of 4, and hence the number of hexadecimal digits in our new N is

$$2\,035\,152 + 504 = 2\,035\,656.$$

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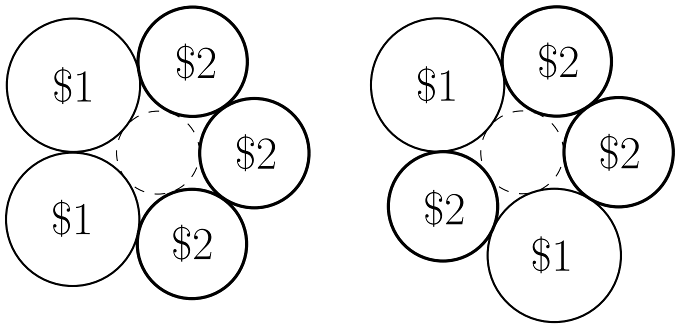
2. An 8-dollar problem

There are just two essentially different ways of arranging three \$2 coins and two \$1 coins in a ring so that each coin is tangent to two others and all five coins are externally tangent to a circle inside the ring.

For which arrangement is the radius of the inner circle larger? See diagram below.



Note. The official diameters of the \$1 and \$2 coins is 25.0 mm and 20.5 mm, respectively, though one may solve this problem knowing only that a \$1 coin is larger than a \$2 coin.

Solution.
Using the law of cosines for the triangle formed by joining the centres of an adjacent \$1 and \$2 coin and the central circle, with θ the angle at the centre of the central circle, we have

$$(r_1 + r_2)^2 = (r + r_1)^2 + (r + r_2)^2 - 2(r + r_1)(r + r_2)\cos\theta,$$

where r_1, r_2, r are the radii of the \$1 and \$2 coins, and central circle, respectively. Thus,

$$\theta = \arccos\Big(1 - \frac{2r_1r_2}{(r + r_1)(r + r_2)}\Big).$$

For the combination with the two \$1 coins together, we have

$$2\pi = \arccos\Big(1 - \frac{2r_1^2}{(r + r_1)^2}\Big) + 2\arccos\Big(1 - \frac{2r_1r_2}{(r + r_1)(r + r_2)}\Big) + 2\arccos\Big(1 - \frac{2r_2^2}{(r + r_2)^2}\Big),$$

while for the combination with the \$1 coins separated,

$$2\pi = 4\arccos\Big(1 - \frac{2r_1 r_2}{(r+r_1)(r+r_2)}\Big) + \arccos\Big(1 - \frac{2r_2^2}{(r+r_2)^2}\Big).$$

With $r_1 = 12.5$ and $r_2 = 10.25$, we find for the first case $r \approx 7.7740$, and for the second case $r \approx 7.7644$. So the arrangement with the \$1 coins together is the one where the radius of the inner circle is larger.

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3. Attacking chess

A rook and a bishop are randomly placed on two different squares of an 8×8 chessboard devoid of any other pieces.

What is the probability that one of the pieces can take the other, according to the rules of chess?

Solution. If the rook is on one of the 4 central squares, it threatens 14 squares (left-right and up-down) and is threatened from 13 squares along diagonals, a total of 27 squares (of the 63 squares not occupied by the rook). This total is reduced to 25 if the rook is on one of the 12 squares surrounding the 4 central squares, 23 for the 20 squares surrounding the preceding 12, and 21 for the 28 squares along the sides of the chessboard. The probability required is therefore

$$\frac{4}{64}\cdot\frac{27}{63} + \frac{12}{64}\cdot\frac{25}{63} + \frac{20}{64}\cdot\frac{23}{63} + \frac{28}{64}\cdot\frac{21}{63} = \frac{13}{36} = 0.36111\ldots$$

The problem may be generalised to an $n \times n$ chessboard, in which case the probability is

$$\frac{10n-2}{3n(n+1)}, n > 1.$$

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4. Fortune chests

Before you are 4 chests, made respectively of iron, copper, teak and ebony. You are told that one chest contains treasure, while the other 3 chests contain poisonous scorpions. You are allowed to choose one chest, lift its lid and plunge your hand inside to take its contents. On each of the 4 lids is inscribed a statement that is either true or false, but you do not know which of them are true.

Here are four statements:

On the iron chest: “If the treasure is in the teak chest, then the statement on the ebony chest is false”.

On the copper chest: “The treasure is in the teak chest or the ebony chest.”

On the teak chest: “One and only one of the four statements is true”.

On the ebony chest: “The statements given in the two metal chests are either both true or both false”.

In which chest can you safely plunge your hand to retrieve the treasure?

Solution. The treasure is in the ebony chest. In order to prove this, let us first denote by *Fe*, *Cu*, T, and E, the statement inscribed on the lids of the iron, copper, teak, and ebony chest, respectively.
First suppose *Fe* is false. Then the treasure would be in the teak chest and E would be true. Hence *Cu* is false (as we already have *Fe* false), so that the treasure is not in the teak chest, contradicting what we had previously deduced.
Therefore, *Fe* is true. But then T is false (otherwise there would be at least two true statements, namely *Fe* and T, a contradiction). So at least one of the two statements *Cu* and E is true. If E is false, then *Cu* would be also (since we already know that *Fe* is true), a contradiction. Therefore, E is true and thus so is *Cu*. Hence, the treasure is in the teak or ebony chest. If the treasure is in the teak, then the true statement *Fe* implies that E is false, a contradiction. And so, the treasure is in the ebony chest.

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5. Functionally sound?

What are all pairs (f, g) of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that f and g never cancel* and that

$$\left(\frac{f}{g}\right)' = \frac{f'}{g'} \text{ and } f'g' = fg?$$

*:the correct word here for *cancel* is *vanish*, i.e. f, g are *never zero*.

Solution. Since f and g never vanish, we have $f(x) \neq 0$ and $g(x) \neq 0$ for all $x \in \mathbb{R}$ and since $f'g' = fg$ we have $f'(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in \mathbb{R}$. Rearranging $f'g' = fg$ we have

$$f = \frac{f'g'}{g}.$$

Also,

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} = \frac{f'}{g'}.$$

Combining these we have

$$\frac{f'g^2 - f'(g')^2}{g^3} = \frac{f'}{g'}$$

$$\therefore \frac{f'}{g'}\left(\frac{g'}{g} - \left(\frac{g'}{g}\right)^3 - 1\right) = 0.$$

Since we deduced that f' and g' never vanish, we have

$$\frac{g'}{g} - \left(\frac{g'}{g}\right)^3 - 1 = 0,$$

which has solution

$$\frac{g'}{g} = C = -18^{-\frac{1}{3}}\left((9-\sqrt{69})^{1/3} + (9+\sqrt{69})^{1/3}\right) \approx -1.3247.$$

Hence $g = Ae^{Cx}$ and $f = Be^{x/C}$ for some constants A and B.

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6. Box of balls

A box contains 1000 blue balls. Outside the box there is an unlimited supply of blue, red and green balls. Every 10 seconds, we choose two balls from the box and replace them by one or two outside balls, using the following replacement rules:

- Two blue with a green;
- Two red with a green;
- Two green with a blue and red;
- One blue and one green with a red;
- One red and one green with a blue;
- One blue and one red with a blue and a red.

Is it possible that the box contains only one ball after a finite time?

Solution.
For each state S of the box, containing b blue balls, g green balls and r red balls, associate the function $f(S) = b + 2g + 3r$. Let S_0 denote the initial state and S_t the state after $10t$ seconds. So we have $f(S_0) = 1000$ and we easily check that, whatever the chosen operation to go from S_t to S_{t+1}, we have $f(S_{t+1}) = f(S_t)$ or $f(S_t) - 4$. Since $f(S_0)$ is a multiple of 4, we deduce that $f(S_t)$ is a multiple of 4 for all $t \in \mathbb{N} \cup \{0\}$. Thus, a state S_t in which there would be a single ball in the box is impossible, since we would have $f(S_t) = 1$, 2 or 3 depending on whether the remaining ball is blue, green or red.

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7. Integer pairs

What are all pairs (m, n) of positive integers such that $2^{2016} + 2^{2012} + 2^{2008} + 2^m = n^2$?

Solution.
The given equation can be rewritten $3 \cdot 7 \cdot 13 \cdot 2^{2008} + 2^m = n^2$. Since 2^m is not divisible by 3, nor is n, i.e. $n \equiv \pm 1 \pmod 3$, and so $n^2 \equiv 1 \pmod 3$. It follows that $2^m \equiv 1 \pmod 3$, and hence m is even since $2 \equiv -1 \pmod 3$. Let $m = 2k$ with integer $k > 0$. Then the equation becomes $3 \cdot 7 \cdot 13 \cdot 2^{2008} = (n - 2^k)(n + 2^k)$. Therefore,

$$n - 2^k = a2^b \tag{1}$$

$$\text{and } n + 2^k = c2^d \tag{2}$$

where $a, b, c, d \in \mathbb{Z}$ such that, $ac = 3 \cdot 7 \cdot 13$ and $b + d = 2008$. By subtracting (1) and (2), we obtain $2^{k+1} = c2^d - a2^b$. Since a and c are odd, it follows that $b = d$ (since with $b \neq d$, on division of both sides by $2^{\min b,d}$, the highest power of 2 dividing $c2^d - a2^b$, one would find that 2^{k+1} has a odd divisor greater than 1, a contradiction).
Thus we have $n - 2^k = a2^{1004}$ and $n + 2^k = c2^{1004}$, where a, c are such that $ac = 3 \cdot 7 \cdot 13$ and $a - c$ is a power of 2. Enumerating the possibilities and checking each in turn, we find that the only solutions are $(a, c) = (39, 7)$ or $(21, 13)$, with $39 - 7 = 2^5$ and $21 - 13 = 2^3$.
Thus we conclude that $(m, n) = (2016, 23 \cdot 2^{1004})$ or $(2012, 17 \cdot 2^{1004})$.

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8. Tangential

What is the value of the product

$$\left(\sqrt{3}+\tan(1^\circ)\right)\left(\sqrt{3}+\tan(2^\circ)\right)\left(\sqrt{3}+\tan(3^\circ)\right)\cdots\left(\sqrt{3}+\tan(28^\circ)\right)\left(\sqrt{3}+\tan(29^\circ)\right)?$$

Solution.
For $n = 1, 2, \ldots, 29$ we have

$$\sqrt{3}+\tan(n^\circ) = \frac{\sin(60^\circ)}{\cos(60^\circ)} + \frac{\sin(n^\circ)}{\cos(n^\circ)} = \frac{\sin(60^\circ+n^\circ)}{\cos(60^\circ)\cos(n^\circ)} = 2\frac{\cos(30^\circ-n^\circ)}{\cos(n^\circ)}.$$

Hence

$$\prod_{n=1}^{29}\left(\sqrt{3}+\tan(n^\circ)\right) = \prod_{n=1}^{29} 2\frac{\cos(30^\circ-n^\circ)}{\cos(n^\circ)} = 2^{29}\frac{\cos(29^\circ)\cdots\cos(1^\circ)}{\cos(1^\circ)\cdots\cos(29^\circ)} = 2^{29}.$$

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9. Continuously decreasing

Is it true, that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous decreasing function, then the system of equations

$$\begin{aligned} x &= f(y), \\ y &= f(z), \\ z &= f(x), \end{aligned}$$

has a unique solution in $\mathbb{R}^3$?

Solution.
Let us first prove that a continuous, decreasing function $f : \mathbb{R} \to \mathbb{R}$ has a unique fixed point. Indeed, the continuous function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(x) - x$ is decreasing (since $x < y$ implies $f(x) > f(y)$, and consequently, $g(x) = f(x) - x > f(y) - y = g(y)$). Moreover, for all $x \geq 0$, we have $g(x) = f(x) - x \leq f(0) - x$, and for all $x \leq 0$, $g(x) = f(x) - x \geq f(0) - x$. It follows that $g(x) \to \mp\infty$, as $x \to \pm\infty$. Thus, by the Intermediate Value Theorem, there exists $x_0 \in \mathbb{R}$ such that $g(x_0) = a$, that is to say, $f(x_0) = x_0$. The function f cannot have another fixed point, because if to the contrary x and x' are two fixed points such that $x < x'$, then $x = f(x) > f(x') = x'$, a contradiction. The triple (x_0, x_0, x_0), where x_0 is the only fixed point of f, is clearly a solution of the system. On the other hand, if (x, y, z) is a solution of the system then $x = f(y) = f(f(z)) = f(f(f(x)))$, and similarly, $y = f(f(f(y)))$ and $z = f(f(f(z)))$. In other words, x, y and z are necessarily fixed points of the continuous function $f^{\circ 3} = f \circ f \circ f$, which is decreasing as $a < b$ implies successively $f(a) > f(b)$, $f(f(a)) < f(f(b))$ and $f(f(f(a))) > f(f(f(b)))$. The $f^{\circ 3}$ function therefore has a unique fixed point, and the fixed point x_0 of f is clearly a fixed point of $f^{\circ 3}$. Hence the triple (x_0, x_0, x_0) is the unique solution of the system. So, indeed, it is true that the system has a unique solution in $\mathbb{R}^3$.

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10. Disjoint circles

In the Euclidean plane, draw three disjoint circles, each exterior to the others, and with their centres noncollinear.

If for each of the three circles, one joins the centre of that circle to the intersection of the interior common tangents to the two other circles, are the three lines obtained necessarily concurrent?

Solution.
Denote the three circles by C, C', C'', and let their centres and radii be O, O', O'', and r, r', r'', respectively. The common internal tangents to C and C' (resp. C' and C'', C'' and C) intersect at a point P'' (resp. P, P'), that is located symmetrically on the line joining the two circle centres. Let T and T' be the points of contact of one of these tangents with C and C', respectively. Since the triangles OTP'' and $O'T'P''$ are similar,

$$\frac{|OP''|}{|O'P''|} = \frac{r}{r'}.$$

Similarly,

$$\frac{|O'P|}{|O''P|} = \frac{r'}{r''} \text{ and } \frac{|O''P'|}{|OP'|} = \frac{r}{r'}.$$

Hence,

$$\frac{|OP''|}{|O'P''|} \cdot \frac{|O'P|}{|O''P|} \cdot \frac{|O''P'|}{|OP'|} = \frac{r}{r'} \cdot \frac{r'}{r''} \cdot \frac{r''}{r} = 1,$$

and therefore, by Ceva's Theorem applied to triangle $OO'O''$, the cevians OP, $O'P'$ and $O''P''$ are concurrent. So, yes, the three lines defined in the problem are necessarily concurrent.