The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2017 Problems with Solutions

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers).

1. Series-ly big

Two sequences of integers are defined by $a_0 = b_0 = 9$, $a_{n+1} = 9^{a_n}$ and $b_{n+1} = b_n!$ for all integers $n \geq 0$.
Which is larger: a_{2017} or b_{2017}?

Solution. Our strategy is to show $a_n > 2b_n \log_9(b_n)$ for all $n \in \mathbb{N}$ by induction, from which it follows that

$$a_{2017} > 2b_{2017} \log_9(b_{2017}) > b_{2017}.$$

Let $P(n) : a_n > 2b_n \log_9(b_n)$.
$P(1)$ holds, since

$$\begin{aligned} a_1 = 9^9 &= 387\,420\,489 \\ &> 6\,531\,840 \\ &= 2 \cdot 9! \cdot 9 \\ &= 2 \cdot 9! \cdot \log_9(9^9) \\ &> 2 \cdot 9! \cdot \log_9(9!) = 2b_1 \log_9 b_1. \end{aligned}$$

If the above inequality is true for an integer $n > 0$, it is also true for $n+1$. Indeed, $a_{n+1} = 9^{a_n} > 9^{2b_n \log_9 b_n} = (b_n)^{2b_n} = (b_n^{b_n})^2 > (b_n!)^2$ and we have to prove that $(b_n!)^2 > 2b_{n+1} \log_9 b_{n+1} = 2(b_n!) \log_9(b_n!)$. That is, $b_n! > 2\log_9(b_n!)$, which is the case since $x > 2\log_9 x$ (or in other words, $9^x > x^2$) for any real number $x > 0$.

---

2. Cutting triangles

Let T be a triangle whose side lengths are 3, 4 and 5.

What is the length of the smallest line segment cutting this triangle into two pieces with the same area?

Solution. Let $T = \triangle ABC$ with $|AB| = 5$, $|AC| = 4$ and $|BC| = 3$. Since $3^2 + 4^2 = 5^2$, ABC is a right-angled triangle of area 6. Let us first suppose that the cutting of ABC into two pieces of area 3 connects two points P and Q in line segments AB and AC, respectively. Let $x = |AQ|$, $y = |AP|$ and $z = |PQ|$.
The area of the triangle APQ is $\frac{1}{2}xy\sin\theta = 3$, where $\theta = \angle BAC$; so $xy = 6/\sin\theta$.
On the other hand,

$$z^2 = x^2 + y^2 - 2xy\cos\theta = (x-y)^2 + 2xy(1-\cos\theta) = (x-y)^2 + \frac{12(1-\cos\theta)}{\sin\theta}.$$

The minimum of z^2 is thus obtained when $x = y$ and, since $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$ this minimum is 4, so that $z = 2$. Since $x = y$, $x^2 = 6/\sin\theta = 10$, it follows that $x = y = \sqrt{10}$.
A similar calculation shows that the length of the smallest segment cutting ABC into two pieces of the same area is $\sqrt{6}$ if it joins line segments AB and BC, and $\sqrt{12}$ if it joins line segments AC and BC. Hence the minimum length sought is 2.

3. Polynomially prime

Is there a polynomial $p(x)$ with integer coefficients such that $p(x) = 2000$ for at least one integer value of x and $p(x) = 2017$ for four other distinct integer values of x?

Solution. Such a polynomial does not exist. Indeed, since $p(x)$ takes the value 2017 for 4 integer values of x, $q(x) = p(x) - 2017$ has at least four distinct integer roots a, b, c and d. Therefore $q(x) = (x-a)(x-b)(x-c)(x-d)r(x)$, where $q(x)$ and $r(x)$ are polynomials with integer coefficients.

Now, if there exists an integer e such that $p(e) = 2000$, then $q(e) = -17$, that is

$$(e-a)(e-b)(e-c)(e-d)r(e) = -17.$$

Consequently, $e-a$, $e-b$, $e-c$ and $e-d$ are 4 divisors of -17. But the only integer divisors of -17 are 1, -1, 17 and -17; so $1 \cdot -1 \cdot 17 \cdot -17 \cdot r(e) = -17$, or $r(e) = -\frac{1}{17}$, a contradiction since $r(e)$ is an integer.

4. Cube assembly

A person assembles n^3 small white $1 \times 1 \times 1$ cubes together to construct a large $n \times n \times n$ cube, then paints a certain number of faces of the large cube, black.

Knowing that there are exactly 218 small cubes having at least one black face, what is the value of n?

Solution. [**Inspired by Conway Li's solution**]

Suppose that f faces of the large (n^3) cube are painted black in such a way that e of its edges are shared by two black faces and v of its vertices are shared by three black faces. Then the number of small cubes having at least one black face is $b = fn^2 - en + v$. We first list the possible triples (f, e, v). Then we check whether the quadratic $b = 218$ has solutions (Number Theory is useful for excluding integer roots). The results are summarised below:

F	E	V	$b = fn^2 - en + v$	Check
1	0	0	n^2	✗ $218 \neq n^2$
2	0	0	$2n^2$	✗ $218 = 2 \cdot 109 \neq 2n^2$
2	1	0	$2n^2 - n$	✗ $218 = 2 \cdot 109 \neq n(2n-1)$
3	2	0	$3n^2 - 2n$	✗ $218 = 2 \cdot 109 \neq n(3n-2)$
3	3	1	$3n^2 - 3n + 1$	✗ $218 \not\equiv 1 \pmod 3$
4	4	0	$4n^2 - 4n$	✗ $218 \not\equiv 0 \pmod 4$
4	5	2	$4n^2 - 5n + 2$	✓ $218 - 2 = 6^3 = 8 \cdot 27 = n(4n-5)\vert_{n=8}$
5	8	4	$5n^2 - 8n + 4$	✗ $218 - 4 = 2 \cdot 107 \neq n(5n-8)$
6	12	8	$6n^2 - 12n + 8$	✓ $218 - 8 = 6 \cdot 5 \cdot 7 = 6n(n-2)\vert_{n=7}$

Hence $n = 7$ or $n = 8$ are the only solutions.

5. Surdly positive

For which real numbers $x > 0$ is the number $\sqrt[3]{3+\sqrt{x}} + \sqrt[3]{3-\sqrt{x}}$ a positive integer?

Solution. We look for the real numbers $x \geq 0$ such that $\sqrt[3]{3+\sqrt{x}} + \sqrt[3]{3-\sqrt{x}} = n$ where n is a positive integer. By cubing both sides and rearranging, we obtain $n^3 = 6 + 3n\sqrt[3]{9-x}$, or $\left(\frac{n^3-6}{3n}\right)^3 = 9 - x$, which implies $x = 9 - \left(\frac{n^2}{3} - \frac{2}{n}\right)^3$.

Now, $9 - \left(\frac{n^2}{3} - \frac{2}{n}\right)^3 \geq 0$ for $0 < n \leq 2\sqrt[3]{3} \approx 2.88$. Therefore the only possible values of n are 1 and 2. The corresponding values of x are $\frac{368}{27}$ and $\frac{242}{27}$, and it is easily verified that these are indeed the solutions to the problem.

---

6. Really coloured

Suppose we colour each point of $\mathbb{R}^3$ red, yellow or blue.
Is it true that one of these colours "will realise all the distances", that is to say for any real number $d > 0$, there will be at least two points of distance d apart, having this colour?

Solution.
One will say that a colour "realises" a distance d if there are two points of $\mathbb{R}^3$ at distance d having this colour. We will prove that one of the three colours achieves all the distances. Let us proceed by contradiction and assume that there are three distances r, y and b such that red does not realise r, yellow does not realise y and blue does not realise b. By changing the names of the colours, it is not restrictive to assume that $r \geq y \geq b > 0$.

Then there exists a sphere S of radius r with no red points on S, either because there are no red points at all or this sphere is centered at a red point. All the points on S are therefore yellow or blue. Since $r \geq b > 0$, S contains two points at distance b. Now, if all points on S were blue, then blue would realise b. Thus S must contain at least one yellow point Y. Consider the sphere S' centered at Y with radius y. The spheres S and S' intersect in a circle which must be entirely blue. An easy computation (using Pythagoras) shows that this circle has radius $R = \sqrt{y^2 - \frac{y^4}{4r^2}}$. It easily follows that $b \leq y \leq 2r$ (this uses the fact that $y \leq r$). Hence there exist two points P and Q on this circle at distance b, so blue realises b, a contradiction.
Remark: the negative of the statement is NOT that there exists a distance x not realised by any of the colours.

---

7. Alice and Bob coin a game

Alice and Bob invented a new game. One hundred coins (of Australian denominations from 5 c to \$2) are randomly placed in a single row. Alice and Bob take turns alternately, choosing a coin from one of the ends of the row, with Alice going first. Play continues until Bob takes the last coin. If the amount of money raised by one of the players is greater than that of their opponent, they have won the game. Of course, neither may win, as in the case, for example, that the 100 coins all have the same value.
Does one of the players have a strategy that never loses, regardless of what their opponent does or the numbers of each denomination of coin, and regardless of how the 100 coins are arranged in the row initially?

Solution.
Alice has a strategy to never lose. To see this, imagine that the 100 coins are coloured alternately in red and blue. Before playing her first shot, Alice calculates the sum R of the values of the red coins and the sum B of the values of the blue coins. We can assume, without loss of generality, that $R \leq B$. Alice will make sure to take all the blue coins. It is always possible because, as the colours of the pieces alternate, if at any moment of the game there is an even number of pieces, the pieces at the two ends are of different colours. Whenever Alice is playing, there remains an even number of coins. She can therefore take a blue coin each time. On the other hand, when Bob plays his two choices are both red.

This strategy applies more generally to any row formed by an even number of coins.
Note that this strategy is not necessarily optimal. For instance, if the coins are 2,1,1,2,1,1 in that order, then this strategy will give a draw, when Alice can actually win. More generally, if $R = B$, Alice can improve her chances by computing (every time it is her turn to play) the sum of the remaining blue and red coins, and swap her strategy from picking the blue coins to picking the red coins if the sum of the remaining red coins is larger than the sum of the remaining blue coins.
Note also that if the total number of coins is odd, neither of the players has a strategy to never lose. For example, if there are only 3 coins (one of 2 dollars surrounded by two 5 cent coins), Bob will win, whatever Alice does. But if two 2-dollar coins surround a 5-cent coin, Alice will win, no matter what Bob does.

---

8. Tetra-luck

A regular tetrahedron T is placed on a horizontal plane. Let F be the face on which T lies. T is rotated around one of the randomly selected three edges of F, until it rests on some other face F' containing this edge. The operation is then repeated with one of the three edges of F', and so on.

What is the probability p_n that after n steps, T again rests on the face F?

Solution. It is clear that $p_0 = 1$ and $p_1 = 0$. The only thing that would prevent T resting on the face F after n steps is when it already rests on this face after $n-1$ steps. The probability that this happens is p_{n-1}; that is, the probability that this does not happen is $1 - p_{n-1}$. If you are in this situation, there is a 1 in 3 chance that T rests on face F in the next step. Therefore, $p_n = \frac{1}{3}(1 - p_{n-1})$, which can be written as $p_n - \frac{1}{4} = -\frac{1}{3}\left(p_{n-1} - \frac{1}{4}\right)$. Therefore $p_{n-1} - \frac{1}{4}$ is a geometric progression with common ratio $-\frac{1}{3}$ and initial term $p_0 - \frac{1}{4} = \frac{3}{4}$. Hence $p_n = \frac{3}{4}\left(-\frac{1}{3}\right)^n + \frac{1}{4}$.

---

9. Working at a two-dimensional ball

In the plane $\mathbb{R}^2$, we consider two concentric discs of radii 1 and $r_n > 1$, where r_n is such that the ring between the two discs contains n discs (the "balls") of diameter $r_n - 1$, each disc being tangent to its two neighbours in the ring.

If A_n denotes the sum of the areas of the n balls and B_n the ring area, what is the limit of A_n/B_n as n tends to infinity?

Solution.
The "balls" are of radius $\rho_n = \frac{1}{2}(r_n - 1)$. So,

$$\frac{A_n}{B_n} = \frac{n\pi\rho_n^2}{\pi r_n^2 - \pi} = \frac{n(r_n - 1)^2}{4(r_n^2 - 1)} = \frac{n(r_n - 1)}{4(r_n + 1)}.$$

Let O be the common centre of the two concentric discs, T the point at which two adjacent "balls" touch, C the centre of one of the two "balls", and $\alpha = \angle COT$. Then, since $\alpha = \pi/n$, we have

$$\sin\left(\frac{\pi}{n}\right) = \sin\alpha = \frac{\rho_n}{1 + \rho_n},$$

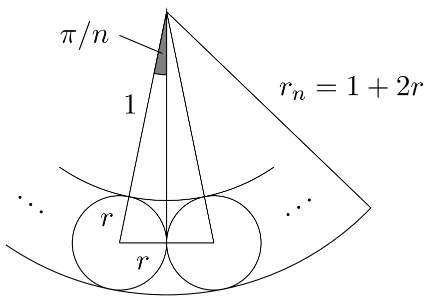
and hence

$$\frac{A_n}{B_n} = \frac{n}{4}\sin\left(\frac{\pi}{n}\right)$$

$$\therefore \lim_{n\to\infty} \frac{A_n}{B_n} = \frac{\pi}{4},$$

since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$



---

10. Octuple triplets

In Euclidean space $\mathbb{R}^3$, does there exist a set S of 8 points such that in each subset of size 3 of S, at least two of the distances between these points are equal?

Solution.
The answer is: Yes. Consider, in the plane Oxy of $\mathbb{R}^3$, a regular pentagon inscribed in a circle of radius 1 centred at the origin. The 5 vertices of the pentagon, the origin $(0, 0, 0)$ and the 2 points $(0, 0, \pm 1)$ form a set S of 8 points such that, in each subset of 3 points of S, at least two of three distances between these points are equal.
Hallard Croft demonstrated in 1962 that there is no set of 9 points of $\mathbb{R}^3$ having this property.