The University of Western Australia
DEPARTMENT OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2018 Problems with Solutions

1. Circular covers

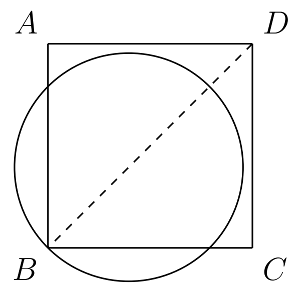
Can one completely cover a square table with sides of length 0.9 metres with two circular tablecloths whose diameters are each 1.006 metres?

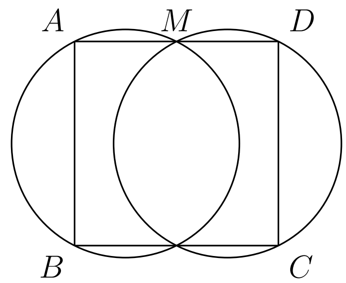
Solution. Let A, B, C, D be the four corners of the table. Then we are given $|AB| = |BC| = |CD| = |DA| = 0.9$. The length of a diagonal of the table is $0.9\sqrt{2} \approx 1.27$, which is larger than the diameter 1.006 of each tablecloth; so two opposite corners cannot be covered by the same tablecloth. Without loss of generality we can then assume that the first tablecloth covers A and B while the second covers C and D. Let M be the middle of the side AD, which needs to be covered by one of the tablecloths. Note that distance from M to B and from M to C is

$$\sqrt{\left(\frac{0.9}{2}\right)^2 + 0.9^2} = 0.9\frac{\sqrt{5}}{2} \approx 1.00623.$$

Hence M cannot be covered by the tablecloth covering B nor the one covering C.

So the answer is: No, a square table of side 0.9 metres cannot be covered by two circular tablecloths whose diameters are each 1.006 metres.





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2. Cos salad

Evaluate $\left(1 - \dfrac{\cos 61^\circ}{\cos 1^\circ}\right)\left(1 - \dfrac{\cos 62^\circ}{\cos 2^\circ}\right) \cdots \left(1 - \dfrac{\cos 119^\circ}{\cos 59^\circ}\right)$.

Solution. First observe that

$$\begin{aligned} 1 - \frac{\cos(60^\circ + k^\circ)}{\cos k^\circ} &= \frac{\cos k^\circ - \cos(60^\circ + k^\circ)}{\cos k^\circ} \\ &= \frac{2\sin 30^\circ \sin(30^\circ + k^\circ)}{\cos k^\circ} \\ &= \frac{\sin(30^\circ + k^\circ)}{\cos k^\circ} \\ &= \frac{\cos(60^\circ - k^\circ)}{\cos k^\circ}. \end{aligned}$$

Hence,

$$\begin{aligned} \left(1 - \frac{\cos 61^\circ}{\cos 1^\circ}\right)\left(1 - \frac{\cos 62^\circ}{\cos 2^\circ}\right) \cdots \left(1 - \frac{\cos 119^\circ}{\cos 59^\circ}\right) &= \frac{\cos 59^\circ}{\cos 1^\circ} \cdot \frac{\cos 58^\circ}{\cos 2^\circ} \cdots \frac{\cos 1^\circ}{\cos 59^\circ} \\ &= 1 \end{aligned}$$

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3. Floating Perth

If the Earth rotated fast enough around its axis, the gravity at a point P of the globe could be counterbalanced by the vertical component of the centrifugal force at P, so that P would be in a state of "zero gravity".

To the closest minute, what would the length of a day be, if Perth were to experience "zero gravity"?

Assume the Earth is a perfect sphere of radius 6 371 km, $g = 9.81\,\mathrm{m/s^2}$, and Perth is at latitude 31.9505°S.

Solution. For Perth to be at zero gravity, it is necessary that the weight mg of a body of mass m is compensated by the vertical component of the centrifugal force,

$$\frac{mv^2}{r} \cdot \cos\lambda,$$

where λ is the latitude of Perth, r is the distance from Perth to the rotational axis of the Earth, and

$$v = \frac{2\pi r}{T},$$

is the velocity. Here T is the period of rotation of the Earth (otherwise known as a day). Since $r = R\cos\lambda$, where R is the radius of the Earth, we have

$$\begin{aligned}
mg &= \frac{mv^2}{r} \cdot \cos\lambda \\
&= \frac{m}{r} \cdot \left(\frac{2\pi r}{T}\right)^2 \cdot \cos\lambda \\
&= m \cdot \left(\frac{2\pi}{T}\right)^2 \cdot r \cdot \cos\lambda \\
&= m \cdot \left(\frac{2\pi}{T}\right)^2 \cdot R \cdot \cos^2\lambda \\
\therefore T^2 &= \big(2\pi\cos\lambda\big)^2 \cdot \frac{R}{g} \\
T &= 2\pi\cos\lambda\sqrt{R/g} \\
&\approx 2\pi\cos(31.9505^\circ)\sqrt{\frac{6\,371 \times 10^3\,\mathrm{m}}{9.81\,\mathrm{m/s^2}}} \\
&\approx 4296.4\,\mathrm{s} \\
&\approx 71.6\,\mathrm{min}
\end{aligned}$$

Thus, Perth would be a point of zero gravity, if the Earth's day was about 72 minutes.

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4. Two spheres

In $\mathbb{R}^3$, let S be a sphere of radius 1 and let S' be a sphere of radius r passing through the centre of S.

For which value(s) of r is the area of the intersection of S' with the interior of S a maximum?

Solution. Denote by $A(r)$ the area of the intersection of S' with the interior of S, and by O, resp. O', the centre of S, resp. S'.
Note first that if $r \leq \frac{1}{2}$, then the whole sphere S' is inside S, so $A(r)$ is just the area of S', that is, $A(r) = 4\pi r^2$. The maximum value in that case is when $r = \frac{1}{2}$, where $A(r) = \pi$.
Assume now that $r > \frac{1}{2}$, so that S and S' intersect in a circle. Let P be a point on this circle and α be the angle between $O'O$ and $O'P$. Taking O' as the origin and $O = (0, 0, r)$, a parameterisation for the intersection of S' with the interior of S is

$$\{(r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r\cos\varphi) \mid \theta \in [0, \pi], \varphi \in [0, \alpha]\}.$$

The normal vector $\partial S/\partial\theta \times \partial S/\partial\varphi$ has norm $r^2 \sin\varphi$; so

$$\begin{aligned} A(r) &= \int_0^\alpha \int_0^{2\pi} r^2 \sin\varphi \, d\theta \, d\varphi \\ &= \int_0^\alpha 2\pi r^2 \sin\varphi \, d\varphi \\ &= 2\pi r^2 (1 - \cos\alpha). \end{aligned}$$

Using the Cosine Rule in triangle $O'OP$, we get that $1 = r^2 + r^2 - 2r^2 \cos\alpha = 2r^2(1 - \cos\alpha)$.
Therefore $A(r) = \pi$.
Thus $A(r)$ is maximal for all $r \geq \frac{1}{2}$.

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5. Functionally composed

We define the iterations of a function f from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$f^1(x) = f(x) \quad \text{and} \quad f^n(x) = f\big(f^{n-1}(x)\big), \text{ for all integers } n > 1.$$

For instance, $f^2(x) = f\big(f(x)\big)$ and $f^3(x) = f\Big(f\big(f(x)\big)\Big)$.
If $f(x) = x^2 + 10x + 20$, what are all the real solutions of the equation $f^{100}(x) = 0$?

Solution. First, we prove by induction the claim that $f^n(x) = (x+5)^{2^n} - 5$.
The claim holds for $n = 1$, since $f(x) = f^1(x) = x^2 + 10x + 20 = (x+5)^2 - 5$.
Now, suppose the claim holds for $n - 1$, that is, $f^{n-1}(x) = (x+5)^{2^{n-1}} - 5$. Then

$$\begin{aligned} f^n(x) &= f\big(f^{n-1}(x)\big) \\ &= f\big((x+5)^{2^{n-1}} - 5\big) \\ &= \Big(\big((x+5)^{2^{n-1}} - 5\big) + 5\Big)^2 - 5 \\ &= \big((x+5)^{2^{n-1}}\big)^2 - 5 \\ &= (x+5)^{2^n} - 5 \end{aligned}$$

Thus, the induction is complete, proving our claim.
So, in particular, the real solutions of the equation $f^{100}(x) = 0$ are x such that $(x+5)^{2^{100}} = 5$, that is $x = -5 \pm 5^{1/2^{100}}$.

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6. Fractal covering

Take a square S. Inscribe within S a circle D, i.e. D is tangent to all four sides of S. Next we draw in each of the four corners of S, a circle tangent to D and to the two sides forming that corner. We continue in this way, infinitely many times: at each step drawing four circles (one in each corner) tangent to the circle drawn in that corner at the previous step and to the two sides forming that corner of S.

What is the ratio of the total area of all the circles we have drawn with the area of S?

Solution. The obvious approach is to calculate the area of all the discs, sum them up and determine the limit to which the sum converges. We can avoid such calculations, reasoning as follows.

Since the ratio of two areas does not depend on the unit of length chosen, we can assumed that the disk D has radius 1, and so S has area 4. Let P be a vertex of the square S, D' the tangent disc to D and both sides of S from P, O the centre of D, O' the centre of D', T the point of tangency of D and D', and r the radius of D'. Since $|OP| = |OT| + |TO'| + |O'P|$, we have $\sqrt{2} = 1 + r + r\sqrt{2}$, and so we obtain

$$\begin{aligned} r &= \frac{\sqrt{2}-1}{\sqrt{2}+1} \\ &= (\sqrt{2}-1)^2 \\ &= 3 - 2\sqrt{2} \end{aligned}$$

Denote by Δ the union of all the discs and by Γ (resp. Γ') the square whose two opposite vertices are O and P (resp. O' and P). By a symmetry argument, the problem posed is equivalent to calculating the area A of $\Delta \cap \Gamma$. If A' denotes the area of $\Delta \cap \Gamma'$, then $A = A' + \frac{1}{4}\pi + \frac{3}{4}\pi r^2$, the last two terms being the area of a quarter of the disc D and the area of three quarters of the disc D'. But $\Delta \cap \Gamma'$ is nothing more than the image of $\Delta \cap \Gamma$ under an homothety with centre P and ratio r. So $A' = Ar^2$. Therefore,

$$\begin{aligned} A &= Ar^2 + \frac{\pi}{4}(1 + 3r^2) \\ (1 - r^2)A &= \frac{\pi}{4}(1 + 3r^2) \\ A &= \frac{\pi}{4} \cdot \frac{1 + 3r^2}{1 - r^2} \end{aligned}$$

Substituting, $r = 3 - 2\sqrt{2}$, after some calculation, we finally obtain $A = \frac{\pi}{8}(3\sqrt{2} - 2) = 0.88068\ldots$.

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7. Prime triangle

An evil wizard has seized 3 logicians Alice, Brian and Cathy, to whom he has administered a powerful sleeping potion. On their awakening, he announces to them that he has written a prime number on each of their foreheads, and that these 3 numbers (not necessarily distinct) are the lengths of the sides of a triangle whose perimeter is also a prime number. Each logician can see the number on the foreheads of the other two, but cannot see the number on their own forehead. With a sardonic smile, the wizard announces that he will release the first logician who guesses the number on their own forehead correctly. Alice sees a 5 on Brian's forehead and a 7 on Cathy's forehead. After a long time, during which the logicians all remain silent, Alice announces that she has deduced the number written on her own forehead.

What is Alice's number and how did she reason?

Solution. Alice's number is 11, and here is her reasoning:

> "By the Triangle Inequality, my number is less than $5 + 7 = 12$, so I must have a 2, 3, 5, 7, or 11. But the perimeter has to be prime so I reject 2 and 3.
>
> If I had a 5, Cathy would see two 5s, so she would deduce her number is a 3 or a 7. Cathy would then realise that if she had a 3, Alice and Brian, seeing a 3 and a 5, would both know they have a 3 or a 5 and that if either had a 3 the other one would know they have a 5. Since neither Alice nor Brian have guessed their number, Cathy would know she has a 7. Since Cathy has not guessed, I don't have a 5.
>
> If I had a 7, Brian would see two 7s, so he would deduce his number is a 3 or a 5. Brian would then realise that if he has a 3, Cathy, seeing a 3 and a 7, would know she has a 7. Since Cathy has not guessed her number, Brian would know he has a 5. Since Brian has not guessed, I don't have a 7.
>
> Therefore I must have the number 11 written on my forehead."

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8. Random complex

We pick at random two distinct complex roots z and z' among the 2018 roots of the equation $z^{2018} = 1$.
What is the probability that $|z + z'| \geq \sqrt{2 + \sqrt{3}}$?

Solution. In the complex plane, the roots of $z^{2018} = 1$ are the vertices of a regular 2018-gon centred at 0: they are $z_k = e^{i2k\pi/2018} = e^{ik\pi/1009}$, for integer k satisfying $-1008 \leq k \leq 1009$. The modulus of the sum of two vertices z and z' is invariant under rotations centred at 0 (in particular under the rotation centred at 0 mapping z' to 1. Thanks to this symmetry argument, the problem is therefore equivalent to calculating the probability that $|z + 1| \geq \sqrt{2 + \sqrt{3}}$, for $z = \cos\theta + i \sin\theta$ chosen randomly from among the 2017 roots not equal to 1. Now,

$$\begin{aligned}
& & \sqrt{2+\sqrt{3}} &\leq |z+1| \\
&\iff & 2+\sqrt{3} &\leq |z+1|^2 \\
& & &= |\cos\theta + 1 + i\,\sin\theta|^2 \\
& & &= (\cos\theta + 1)^2 + \sin^2\theta \\
& & &= 2 + 2\cos\theta \\
&\iff & \frac{\sqrt{3}}{2} &\leq \cos\theta \\
&\iff & |\theta| &\leq \frac{\pi}{6}.
\end{aligned}$$

Thus $z = z_k \neq 1$ and $|\theta| \leq \pi/6$ implies

$$0 \neq |\theta| = |k|\frac{\pi}{1009} \leq \frac{\pi}{6},$$

which implies $1 \leq |k| \leq \lfloor 1009/6 \rfloor = 168$. Thus $2 \cdot 168 = 336$ of the 2017 roots z_k that are not equal to 1, are such that $|z_k + 1| \geq \sqrt{2 + \sqrt{3}}$. Thus the required probability is $336/2017 = 0.16658\ldots$.

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9. Coincidentally

Given a tetrahedron T in $\mathbb{R}^3$, let S be the centre of the inscribed sphere, S' the centre of the circumscribed sphere, and G the centroid of T. Recall that, if A, B, C, D are the vertices of T, then the centroid G of T, is the point for which

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} = \overrightarrow{0}.$$

Is it necessarily true, that if S, S' and G coincide, then all the edges of T have the same length, so that T is a regular tetrahedron?

Solution. If the centres of the inscribed and circumscribed spheres of a tetrahedron T coincide, it is not necessary for T to be regular. Here is an infinite family of counterexamples.

Let a, b, c be three positive real numbers and suppose the vertices of the tetrahedon T are (a, b, c), $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$ in $\mathbb{R}^3$. Observe that the pairs of opposite edges of T all have the same length, and T is only regular if $a = b = c$.

We will prove that, for each member of this infinite family, the points S, S' and G of T coincide. Here is a very simple symmetry argument that allows us to avoid all calculations. Each of the 180° rotations about the x-, y- and z- axes preserves all four vertices of T. The points S, S' and G of T are determined by the data of the vertices of T, and hence are necessarily fixed by each of these rotations, and so they all coincide with the origin 0 of $\mathbb{R}^3$. Thus, it is not necessarily true, that if S, S' and G coincide, then all the edges of T have the same length.

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10. Consecutive throws

A coin that is weighted in such a way that the probabilities of getting a tail and a head are $p = \frac{1}{3}$ and $q = \frac{2}{3}$, respectively, is thrown infinitely many times.
What is the probability that the first occurrence of 4 consecutive heads is before the first occurrence of 3 consecutive tails?

Solution. Define a sequence to be *good* if the first occurrence of HHHH is before the first occurrence of TTT. Then, all good sequences are of one of the following types:

- The sequence starts with T or TT followed by a good sequence starting with H.
- The sequence starts with H or HH or HHH followed by a good sequence starting with T.
- The sequence starts with HHHH.

Let P_T and P_H be the probabilities of getting a good sequence starting with a tail and a head, respectively. Then

$$\begin{aligned} P_T &= (p + p^2)P_H \\ P_H &= (q + q^2 + q^3)P_T + q^4 \\ \therefore P_H &= (p + p^2)(q + q^2 + q^3)P_H + q^4 \\ \big(1 - (p + p^2)(q + q^2 + q^3)\big)P_H &= q^4 \\ P_H &= \frac{q^4}{1 - (p + p^2)(q + q^2 + q^3)} \\ P_T &= \frac{(p + p^2)q^4}{1 - (p + p^2)(q + q^2 + q^3)} \end{aligned}$$

Thus, the probability of getting a good sequence is

$$\begin{aligned} P_H + P_T &= \frac{(1 + p + p^2)q^4}{1 - (p + p^2)(q + q^2 + q^3)} \\ &= \tfrac{16}{21} \\ &\approx 76.2\% \end{aligned}$$