The University of Western Australia
DEPARTMENT OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2019 Problems with Solutions

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers).

1. Distantly complex

We randomly choose two real numbers b and c (not necessarily distinct) in the interval $[0, 1]$. What is the probability that the roots of the equation

$$z^2 + bz + c = 0$$

are a distance of at most 1 apart, in the complex plane?

Solution. The distance between roots z_1 and z_2 of the equation is the modulus of their difference,

$$|z_1 - z_2| = |\sqrt{b^2 - 4c}| = \sqrt{|b^2 - 4c|}.$$

Thus,

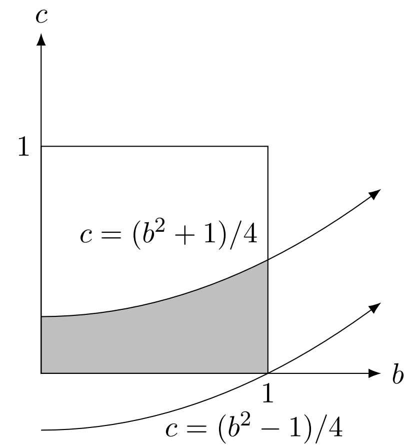
$$\begin{aligned} |z_1 - z_2| \le 1 &\iff -1 \le b^2 - 4c \le 1 \\ &\iff 4c - 1 \le b^2 \text{ and } b^2 \le 4c + 1 \\ &\iff c \le \frac{b^2+1}{4} \text{ and } \frac{b^2-1}{4} \le c \\ &\iff \frac{b^2-1}{4} \le c \le \frac{b^2+1}{4}. \end{aligned}$$

Thus the required probability, is the conditional probability that $(b, c) \in [0, 1] \times [0, 1]$ also lies between the two parabolas

$$c = \frac{b^2-1}{4} \text{ and } c = \frac{b^2+1}{4},$$

which is simply the area of the grey region, i.e.

$$\begin{aligned} \int_0^1 \frac{b^2+1}{4}\, db &= \left[\frac{1}{4}\left(\frac{b^3}{3} + b\right)\right]_0^1 \\ &= \tfrac{1}{3}. \end{aligned}$$



Thus the probability that roots of $z^2 + bz + c = 0$ are at most a distance 1 apart is $\frac{1}{3}$.

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2. Diagonally diverse

In the Euclidean plane $\mathbb{R}^2$, let P be a convex polygon (not necessarily regular) having 21 sides. If we draw all the diagonals of P (that is to say all the lines joining two vertices of P not connected by one side), at least two of these diagonals will form an angle whose measurement is less than $1°$.
True or false?

Solution. The statement is true. Since P is convex, the number of diagonals of P is equal to the number of non-adjacent vertex pairs, namely $(21 \cdot 18)/2 = 189$.
If there are two parallel diagonals, they make an angle of $0° < 1°$, and the required property is satisfied.
Otherwise, the 189 diagonals all have different directions. So, if we draw a line parallel to each of them, that pass through the centre of a fixed circle C, then these 189 lines will cut C in 378 distinct points. Thus, if C is partitioned into 360 semi-open arcs of $1°$, by the Pigeon Hole Principle, since $378 > 360$, two of the aforementioned points will lie in the same arc, so that as a consequence, the required property is again satisfied.

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3. cos-sec salad

What is the following product? (i.e. simplify the expression as much as is possible.)

$$\begin{aligned}\prod_{n=4}^{2019} &\big(2\cos(2^{n\,\circ}) - \sec(2^{n\,\circ})\big)\\ &= \big(2\cos(16°) - \sec(16°)\big)\big(2\cos(32°) - \sec(32°)\big)\cdots\big(2\cos(2^{2019°}) - \sec(2^{2019°})\big).\end{aligned}$$

Solution. Since,

$$\begin{aligned}2\cos\theta - \sec\theta &= \frac{2\cos^2\theta - 1}{\cos\theta}\\ &= \frac{\cos(2\theta)}{\cos\theta},\end{aligned}$$

whenever $\cos\theta \neq 0$, i.e. for $\theta \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$, we have

$$\begin{aligned}\prod_{n=4}^{2019}\big(2\cos(2^{n\,\circ}) - \sec(2^{n\,\circ})\big) &= \prod_{n=4}^{2019}\frac{\cos(2^{n+1\circ})}{\cos(2^{n\,\circ})}\\ &= \frac{\cos(2^{2020°})}{\cos(16°)}.\end{aligned}$$

Now, $2^{2020} - 2^4 = 2^4(2^{2016} - 1)$ and 2016 is divisible by each of 4 and 6. Hence, $2^{2016} - 1$ is divisible by

$$\begin{aligned}\operatorname{lcm}(2^4 - 1, 2^6 - 1) &= \operatorname{lcm}(15, 63)\\ &= 5 \cdot 7 \cdot 9,\end{aligned}$$

so that $2^4(2^{2016} - 1)$ is a multiple of $360°$, and hence $\cos(2^{2020°}) = \cos(16°)$, and thus the required product is 1.

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4. Maximum product

In the Euclidean plane $\mathbb{R}^2$, let P be a regular polygon of n vertices, inscribed in a circle C of radius r.

For which points X of C is the product of the distances from X to the vertices of P maximised?

Solution. We can locate the vertices of P at the complex numbers $r\varepsilon^k$ where $\varepsilon = \mathrm{cis}(2\pi/n)$ and let a point X of C be rz where $|z| = 1$. The product of the distances from X to the vertices of P is then

$$\begin{aligned}\prod_{k=1}^{n} |rz - r\varepsilon^k| &= r^n \prod_{k=1}^{n} |z - \varepsilon^k| \\ &= r^n |z^n - 1| \\ &\le r^n(|z^n| + 1) = 2r^n.\end{aligned}$$

Thus the maximum value of this product is $2r^n$, which is attained if and only if $|z^n - 1| = 2$, which occurs, iff $z^n = -1$. The locus of points X being sought are therefore the midpoints of the arcs of C joining pairs of consecutive vertices of P.

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5. Irractible distraction

A set E of points of Euclidean space $\mathbb{R}^n$ is said to be **irractible** (not to be confused with *irascible*, which is a nasty condition), if every point of $\mathbb{R}^n$ is at an irrational distance from at least one point of E.

If $I(n)$ denotes the smallest number of points in an irractible set of $\mathbb{R}^n$, what is the value of $I(n)$ for each $n \in \mathbb{N}$?

Solution. First observe that $I(n) > 1$, for all $n > 0$. Indeed, for an irractible set of $\mathbb{R}^n$ to reduce to a single point $a \in \mathbb{R}^n$ is impossible, since there is at least one point of $\mathbb{R}^n$ at distance 1 from a. $I(1) = 2$, since $\{0, \sqrt{2}\}$ forms an irractible set of $\mathbb{R}$. Indeed, any irrational number is an irrational distance from 0 and each rational number r is an irrational distance from $\sqrt{2}$ (since $|\sqrt{2} - r| = r' \in \mathbb{Q}$ implies $\sqrt{2} = r \pm r' \in \mathbb{Q}$, contradicting the irrationality of $\sqrt{2}$).
On the other hand, we will show $I(n) \ge 3$ for all $n > 1$. Suppose to the contrary $I(n) = 2$. Then there exists an irractible set $\{a, b\}$. Let r be a rational number greater than half the distance between a and b; then the two spheres of radius r centred at a and b have a non-empty intersection and all points of this intersection are at a rational distance r from a and b (a contradiction). Hence $I(n) \ge 3$ for $n > 1$.
We now prove that $I(n) = 3$ for all $n > 1$. Let α be a real number whose square is irrational (for example, we could take $\alpha = \pi$, or more simply $\alpha = \sqrt[4]{\pi}$). We will now show that

$$\{(-\alpha, 0, \dots, 0), (0, 0, \dots, 0), (\alpha, 0, \dots, 0)\}$$

is an irractible set of $\mathbb{R}^n$. The existence of a point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ at a rational distance from of each of these 3 points, implies

$$(x_1 + \alpha)^2 + x_2^2 + \cdots + x_n^2 \in \mathbb{Q} \tag{1}$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 \in \mathbb{Q} \tag{2}$$
$$(x_1 - \alpha)^2 + x_2^2 + \cdots + x_n^2 \in \mathbb{Q} \tag{3}$$

but (1) $-$ 2(2) + (3) implies $2\alpha^2 \in \mathbb{Q}$ and hence $\alpha^2 \in \mathbb{Q}$, contradicting the choice of α. Thus $I(n) = 3$ for $n > 1$.
Therefore, $I(1) = 2$ and $I(n) = 3$, for $n > 1$.

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6. n-tertaining sequences

For each $n \in \mathbb{N}$, define a sequence whose initial element is n. Then form the next element, by dividing by 3, if the element is a mutiple of 3, or add 10, and repeat this rule for subsequent elements. For example, if $n = 16$, we obtain

$$16,\ 26,\ 36,\ 12,\ 4,\ 14,\ 24,\ 8,\ 18,\ 6,\ 2,\ 12,\ 4,\ 14, \ldots.$$

Observe that the example sequence becomes periodic (of period 8) starting from the 4^{th} term. For what $n \in \mathbb{N}$ is the associated sequence periodic from a certain term on, and what is the period of this "sequel" of the sequence?

Solution. Let $n = x_1, x_2, \ldots, x_i, \ldots$ be the sequence associated with integer $n > 0$.
We first show that if a term x_i of the sequence is greater than 10, then there exists a term $x_j < x_i$ with $j > i$. Indeed,
if $x_i \equiv 0 \pmod 3$, then $x_{i+1} = x_i/3 < x_i$;
if $x_i \equiv 1 \pmod 3$, then $x_{i+3} = (x_i + 20)/3 < x_i$; and,
if $x_i \equiv 2 \pmod 3$, then $x_{i+2} = (x_i + 10)/3 < x_i$.
Consequently, the sequence must eventually contain a term that is less than or equal to 10.
On the other hand, the sequences

$$1, 11, 21, 7, 17, 27, 9, 3, 1, \ldots \text{ (period 8)}$$
$$2, 12, 4, 14, 24, 8, 18, 6, 2, \ldots \text{ (period 8)}$$
$$10, 20, 30, 10, \ldots \text{ (period 3)}$$
$$5, 15, 5, \ldots \text{ (period 2)}$$

show all integers from 1 to 10.
So, the sequence associated with an integer $n > 0$, becomes periodic from a certain term on.
Let $\rho(n)$ be the period of the sequence associated with an integer n.
Observe that x_i ends in 0 (resp. 5) if and only if x_{i+1} ends in 0 (resp. 5).
This implies that the sequence associated with an integer n will become periodic of period 3 if n ends with 0, of period 2 if n ends with 5, and of period 8 in all other cases, i.e.

$$\rho(n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod{10}, \\ 2 & \text{if } n \equiv 5 \pmod{10}, \text{ or} \\ 8 & \text{otherwise.} \end{cases}$$

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7. Well-proportioned triangle

In the Euclidean plane $\mathbb{R}^2$, we denote by A, B, C the measurements of the angles of a triangle and by a, b, c, respectively, the lengths of the sides opposite these angles.
Given that

$$\frac{\cos A}{2} = \frac{\cos B}{9} = \frac{\cos C}{12},$$

what are the proportions of the triangle, i.e. what is $a : b : c$?

Solution. The hypothesis on A, B, C implies the existence of a real number k such that

$$\cos A = 2k, \ \cos B = 9k \text{ and } \cos C = 12k.$$

Since one angle of a triangle is necessarily acute, and therefore has positive cosine, A, B, C each has a positive cosine and is thus acute. So $\cos C < 1$, and consequently $0 < k < \frac{1}{12}$. Since A, B, C are between 0 and $\pi/2$, their sines are positive. Thus

$$\sin A = \sqrt{1 - 4k^2}, \ \sin B = \sqrt{1 - 81k^2} \text{ and } \sin C = \sqrt{1 - 144k^2}.$$

Since $A + B + C = \pi$, we have $\cos(A + B) = -\cos C$, and hence

$$\cos A \cos B - \sin A \sin B + \cos C = 0,$$

from which we deduce (after computations) that

$$432k^3 + 229k^2 - 1 = 0 \qquad (*)$$

Since cubics are in general difficult to solve, lets check to see if there is a rational solution. If $k = m/n$, where m, n are positive coprime integers, then

$$432m^3 + 229m^2 n = n^3,$$

so that $m \,|\, n$, and hence $m = 1$. Thus,

$$432 + 229n = n^3,$$

and hence $n \,|\, 432 = 2^4 \cdot 3^3$ with $n > 12$ (since $k < 12$). We find that $n = 16$ is the only solution possible, and so $(*)$ admits $k = \frac{1}{16}$ as a rational solution, and has two other irrational solutions, $(-8 - \sqrt{37})/27$ and $(-8 + \sqrt{37})/27$. Excluding the irrational solutions, since they are negative, we are fortuitously left with only the rational solution $k = \frac{1}{16}$, and so

$$\sin A = \frac{6\sqrt{7}}{16}, \ \sin B = \frac{5\sqrt{7}}{16} \text{ and } \sin C = \frac{4\sqrt{7}}{16}.$$

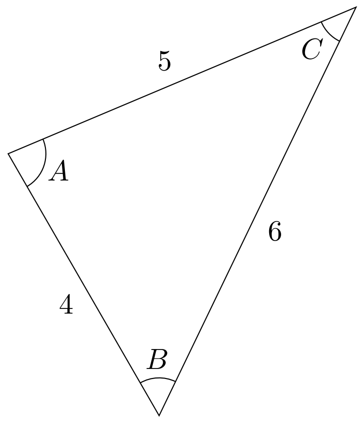
So, finally, applying the Sine Rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

we have

$$a : b : c = 6 : 5 : 4.$$

Drawn to scale the solution triangle is as follows



8. Winning resolution

A row of n empty boxes is drawn on a sheet of paper. Alice and Bob then play a game. In turn, each of them writes one of the two letters "A" or "H" in an empty box. The winner is the first to complete the writing of "AHA" in 3 adjacent boxes, as shown in the following diagram.

	A	H	A	

For which values of n, greater than 2, does Alice, who plays first, have a winning strategy, regardless of what Bob does?

Solution. Let the boxes of the row be numbered in order $c_1, c_2, \ldots, c_n$.
If $n = 2k$ is even, let us partition the set of n boxes in k pairs $\{c_{2i-1}, c_{2i}\}$, for $i = 1, \ldots, k$. If each time Alice writes a letter in a box of one of these pairs, Bob decides to write the same letter in the other box, the word AHA will never appear in this case, because any letter will have at least one identical neighbour, which is not the case for the H in AHA. The game is thus null (no winner), so that Alice does not have a winning strategy.
Suppose now that n is odd. For $n = 3$ and $n = 5$, a simple case study shows that, if Alice and Bob play optimally, any game is null.
From now on, we therefore assume odd $n \geq 7$. An empty box will be called a *loser box* if, a player writing a letter in this box, allows their opponent to win on the next move. Thus, if one writes an H in a loser box, since the opponent will be able to complete AHA on the next move, the loser box is adjacent to an A on one side and another empty box on the other side. Similarly, if one writes an A in a loser box, since the opponent will be able to complete AHA on the next move, and since any loser box is between an A and an empty box, the latter (empty box) must also be adjacent to an A (so that the opponent can win by putting an H) and, by symmetry, this empty box is also a losing box. The losing boxes are therefore grouped in pairs (surrounded by two As), as shown in the diagram.

	A			A	

Since $n \geq 7$, Alice can build a pair of losing boxes in two moves at the beginning of the game. For instance, on her first move, she writes an A in c_4. On her second move, if she can win (because Bob wrote an H next to this A), she of course takes advantage of the situation. Otherwise, if Bob plays in c_i with $i < 4$, Alice puts an A in c_7, or in c_1, if Bob plays in a different manner.
As n is odd, every time it's Alice's turn to play, the number of empty boxes is odd and when it's Bob's turn, the number of empty boxes number is even. Consequently, Alice is sure to win because there is always at least one empty non-losing box when it is her turn to play (Alice must of course seize any opportunity to win if Bob plays badly and offers her victory). Sooner or later, Bob will only have losing boxes, since Alice could build at least two, and so he will lose the game.
In conclusion, Alice has a winning strategy if and only if n is at least 7 and odd.

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9. Fractionally continuous

Given $$a = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\ddots + \cfrac{1}{2018}}}} \quad \text{and} \quad b = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\ddots + \cfrac{1}{2018 + \cfrac{1}{2019}}}}},$$

prove that $|a - b| < \dfrac{1}{2018!\,2019!}$.

Solution. Let

$$a_n = \cfrac{1}{n + 2 + \cfrac{1}{n + 3 + \cfrac{1}{\ddots + \cfrac{1}{2018}}}} \quad \text{and} \quad b_n = \cfrac{1}{n + 2 + \cfrac{1}{n + 3 + \cfrac{1}{\ddots + \cfrac{1}{2018 + \cfrac{1}{2019}}}}}.$$

Then $a = a_0$ and $b = b_0$. Observe that

$$\begin{aligned}
a_n &= \frac{1}{n + 2 + a_{n+1}} \quad \text{and} \quad b_n = \frac{1}{n + 2 + b_{n+1}} \\
\therefore |a_n - b_n| &= \left| \frac{1}{n + 2 + a_{n+1}} - \frac{1}{n + 2 + b_{n+1}} \right| \\
&= \left| \frac{n + 2 + b_{n+1} - (n + 2 + a_{n+1})}{(n + 2 + a_{n+1})(n + 2 + b_{n+1})} \right| \\
&= \frac{|b_{n+1} - a_{n+1}|}{(n + 2 + a_{n+1})(n + 2 + b_{n+1})} \\
&< \frac{|a_{n+1} - b_{n+1}|}{(n + 2)^2}.
\end{aligned}$$

Thus, inductively we have,

$$\begin{aligned}
|a-b| &= |a_0 - b_0| \\
&< \frac{1}{2^2} \cdot |a_1 - b_1| \\
&< \frac{1}{2^2 \cdot 3^2} \cdot |a_2 - b_2| \\
&\vdots \\
&< \frac{1}{2^2 \cdot 3^2 \cdots k^2} \cdot |a_{k-1} - b_{k-1}| \\
&< \frac{1}{2^2 \cdot 3^2 \cdots 2017^2} \cdot |a_{2016} - b_{2016}| \\
&= \frac{1}{2^2 \cdot 3^2 \cdots 2017^2} \cdot \left| \frac{1}{2018} - \frac{1}{2018 + 1/2019} \right| \\
&= \frac{1}{2^2 \cdot 3^2 \cdots 2017^2} \cdot \frac{1/2019}{2018(2018 + 1/2019)} \\
&< \frac{1}{2^2 \cdot 3^2 \cdots 2017^2} \cdot \frac{1/2019}{2018 \cdot 2018} \\
&= \frac{1}{2018!\,2019!}
\end{aligned}$$

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10. Continuously even-to-one

Is there a continuous function $f : [0,1] \to \mathbb{R}$ such that, for every $r \in \mathbb{R}$, the number of solutions $x \in [0,1]$ of the equation $f(x) = r$ is an even non-negative integer?

Solution. The following is an example (among others) of a continuous function $f : [0,1] \to \mathbb{R}$ such that, for all $r \in \mathbb{R}$, the number of solutions $x \in [0,1]$ for the equation $f(x) = r$ is an even non-negative integer.
In the plane $\mathbb{R}^2$, define the points

$$a_0 = (1,0),\ b_0 = (\tfrac{1}{2}, \tfrac{1}{2}),\ a_k = (2^{-2k}, 2^{-(k+1)}),\ b_k = (2^{-(2k+1)}, 2^{-k}),\ \text{for } k \geq 1.$$

The graph of f is the union of closed segments $[a_k, b_k]$ and $[b_k, a_{k+1}]$ for all $k \geq 0$ and we also define $f(0) = 0$. One can show that the piecewise linear function thus defined is continuous and any straight line parallel to the x-axis intersects the graph of f in 0, 2 or 4 points. We could slightly modify this example to obtain a differentiable function having the same property.
