The University of Western Australia
DEPARTMENT OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2020 Problems with Solutions

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers).

1. Equilaterally so

Let x be the side length of an equilateral triangle ABC, and suppose points P and Q lie in the interior of ABC such that $PQ = 1$, $AP = AQ = \sqrt{7}$ and $BP = CQ = 2$, where line segments BP and CQ do not intersect.
What is the value of x?

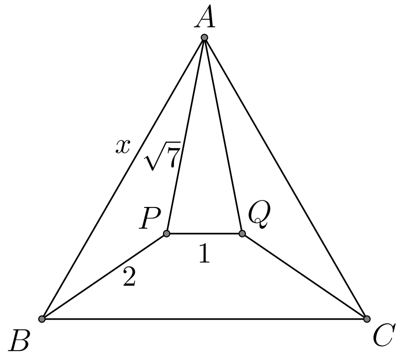
Solution. Since $AB = AC$, $AP = AQ$, $BP = CQ$,

$$\triangle ABP \cong \triangle ACQ, \text{ by the SSS Rule,}$$

from which it follows that the median from A to the midpoint of BC, is an axis of symmetry for the figure.
Let $\alpha = \angle PAQ$ and $\beta = \angle BAP$, and note that

$$\alpha + 2\beta = 60^\circ,$$

so that, in particular, α and β are acute, and hence their sines and cosines are positive.



Applying the Cosine Rule in $\triangle PAQ$, we have

$$\begin{aligned}
1 &= (\sqrt{7})^2 + (\sqrt{7})^2 - 2(\sqrt{7})^2 \cos\alpha \\
\cos\alpha &= \frac{13}{14} \\
\sin\alpha &= \sqrt{1 - (13/14)^2} \\
&= \frac{\sqrt{27}}{14} \\
\cos(2\beta) &= \cos(60^\circ - \alpha) \\
&= \cos 60^\circ \cos\alpha + \sin 60^\circ \sin\alpha \\
&= \frac{1}{2}\cdot\frac{13}{14} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{27}}{14} \\
&= \frac{11}{14} \\
&= 2\cos^2\beta - 1, \text{ (identity for } \cos(2\beta)) \\
\cos^2\beta &= \frac{25}{4\cdot 7} \\
\cos\beta &= \frac{5}{2\sqrt{7}}.
\end{aligned}$$

So now we can apply the Cosine Rule in $\triangle APB$:

$$\begin{aligned} 2^2 &= (\sqrt{7})^2 + x^2 - 2\sqrt{7}x\cos\beta \\ &= 7 + x^2 - 2\sqrt{7}x \cdot \frac{5}{2\sqrt{7}} \\ 0 &= x^2 - 5x + 3. \end{aligned}$$

So taking the positive root of the final equation we have

$$x = \frac{5+\sqrt{25-4\cdot 3}}{2} = \frac{5+\sqrt{13}}{2}.$$

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2. Empowered reciprocals

Let x, y, z be non-zero real numbers such that $x+y+z \neq 0$. If

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z},$$

for which integers $n > 1$, can we deduce that

$$\frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n} = \frac{1}{(x+y+z)^n}?$$

Solution. Assume $x, y, z, x+y+z \neq 0$ and $1/x + 1/y + 1/z = 1/(x+y+z)$. Then

$$\begin{aligned} (xy+yz+zx)(x+y+z) &= xyz \\ (x+y)(y+z)(z+x) &= 0. \end{aligned}$$

Thus one of $x+y$, $y+z$, or $z+x$ is 0. Since the problem is fully symmetric with respect to x, y, z, we may assume without loss of generality that $x+y=0$. Consequently,

$$\frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n} = \frac{1}{(x+y+z)^n} \tag{$*$}$$

reduces to

$$\begin{aligned} \frac{1}{x^n} + \frac{1}{(-x)^n} + \frac{1}{z^n} &= \frac{1}{z^n} \\ x^n + (-x)^n &= 0, \end{aligned}$$

which (since $x \neq 0$) holds if and only if n is odd, i.e. the integers $n > 1$ for which $(*)$ holds are the odd integers $n \geqslant 3$.

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3. Really covered

The set of all points of the plane $\mathbb{R}^2$ whose coordinates are both rational is denoted by $\mathbb{Q}^2$. Does the set of all intersection points of line segments that join two points of $\mathbb{Q}^2$, cover the plane $\mathbb{R}^2$?

Solution. We use the following terminology. In $\mathbb{R}^2$, a *rational point* denotes a point for which both coordinates are rational, and a *rational segment* denotes a segment whose two endpoints are both rational points.
We show that $P = (\sqrt{2}, \sqrt{3})$ is not covered by any rational segment. We first observe that, since $\sqrt{2}$ and $\sqrt{3}$ are irrational, P is not covered by any horizontal or vertical rational segment. Suppose, for a contradiction, P is covered by an oblique rational segment s. Then s is contained in some line with equation $y = ax + b$ where $a, b \in \mathbb{Q}$ and $a \neq 0$. Substituting P, we have

$$\begin{aligned}\sqrt{3} &= a\sqrt{2} + b \\ 3 &= 2a^2 + b^2 + 2a\sqrt{2}\end{aligned}$$

from which we have $\sqrt{2} = (3 - 2a^2 + b^2)/(2a) \in \mathbb{Q}$ (contradiction).
Thus the set of all intersection points of line segments that join two points of $\mathbb{Q}^2$, does not cover $\mathbb{R}^2$.
Note. Using Measure Theory, it is possible to prove much more. Indeed, since $\mathbb{Q}$ is countable, as is $\mathbb{Q}^4$, the set of rational segments is countable. Since the measure of any segment is zero, the measure of a countable union of rational segments is also zero, and so the intersections of rational segments cannot cover $\mathbb{R}^2$, $\mathbb{R}^2$ having non-zero measure. In other words, almost all the points of $\mathbb{R}^2$ (that is, all except those of a subset of measure zero) are not covered!

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4. Coin toss salad

Alice and Bob toss a fair coin many times. If a head appears, Alice gives Bob \$1, and if a tail appears, Bob gives Alice \$1. Initially, Alice has \$$a$, and Bob has \$$b$.
If the game continues until one of the two players has lost everything, what is the probability that Alice wins?

Solution. Let p_n denote the probability that Alice wins the game if she has \$$n$ after a coin toss. Since there are two ways Alice may have \$$n$ after a coin toss: she may have gained or lost \$1, we have

$$p_n = \begin{cases} \frac{1}{2}p_{n+1} + \frac{1}{2}p_{n-1}, & \text{if } n > 0, \\ 0, & \text{if } n = 0. \end{cases}$$

So for $n > 0$, $p_{n+1} - 2p_n + p_{n-1} = 0$, or on rearrangement

$$p_{n+1} - p_n = p_n - p_{n-1},$$

which applied recursively gives:

$$p_{n+1} - p_n = p_n - p_{n-1} = p_{n-1} - p_{n-2} = \cdots = p_2 - p_1 = p_1.$$

Consequently, $p_n = np_1$, for all $n > 0$, and since $p_{a+b} = 1$, we have $p_1 = 1/(a + b)$.
Thus the probability that Alice wins is $p_a = a/(a + b)$.

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5. Sum functions making a difference

Determine all nonconstant, infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) - f(x-y) = f'(x)f'(y),$$

for all $x, y \in \mathbb{R}$.

Solution. We have, by hypothesis,

$$f(x+y) - f(x-y) = f'(x)f'(y) \tag{$*$}$$

for all $x, y \in \mathbb{R}$.
By setting $x = y = 0$, we obtain $f'(0) = 0$.
Differentiating $(*)$ with respect to y, three times in succession, we obtain

$$\begin{aligned} f'(x+y) + f'(x-y) &= f'(x)f''(y) \\ f''(x+y) - f''(x-y) &= f'(x)f'''(y) \\ f'''(x+y) + f'''(x-y) &= f'(x)f''''(y), \end{aligned}$$

and now setting $y = 0$, we have

$$2f'(x) = f'(x)f''(0) \tag{1}$$
$$0 = f'(x)f'''(0) \tag{2}$$
$$2f'''(x) = f'(x)f''''(0) \tag{3}$$

for all $x \in \mathbb{R}$. As f is assumed to be non-constant, there exists a real number x^* such that $f'(x^*) \neq 0$. Setting $x = x^*$ in (1) and (2), we get $f''(0) = 2$ and $f'''(0) = 0$. The functions f sought are therefore solutions of the differential equation (3), with the initial conditions $f'(0) = 0$, $f''(0) = 2$ and $f'''(0) = 0$.
The condition $f'''(0) = 0$ will turn out to be redundant. Let $u(x) = f'(x)$ and let $\lambda = f''''(0)/2$. Then our differential equation with initial conditions becomes:

$$u'' - \lambda u = 0,\ u(0) = 0, u'(0) = 2. \tag{$**$}$$

Now let $\mathsf{D} = d/dx$, and note the following lemma.

Lemma. $e^{\alpha x}$ *is a solution of* $(\mathsf{D} - \alpha)u = 0$.

Proof. The result follows immediately, by direct substitution:

$$\begin{aligned} (\mathsf{D} - \alpha)(e^{\alpha x}) &= \alpha e^{\alpha x} - \alpha e^{\alpha x} \\ &= 0. \end{aligned}$$

□

We note that a homogeneous constant coefficient linear differential equation has the properties:

(a) If a differential operator L factorises as

$$\mathsf{L} = (\mathsf{D} - \alpha_1)(\mathsf{D} - \alpha_2)\cdots(\mathsf{D} - \alpha_k)$$

where $\alpha_1, \ldots, \alpha_k$ are distinct constants, then each of $e^{\alpha_1 x}, \ldots, e^{\alpha_k x}$ is a solution of $\mathsf{L}\,u = 0$.

(b) If $e^{\alpha_1 x}, \dots, e^{\alpha_k x}$ are solutions of $\mathsf{L}\,u = 0$ then any linear combination of $e^{\alpha_1 x}, \dots, e^{\alpha_k x}$ is also a solution of $\mathsf{L}\,u = 0$.

If there are no other independent solutions of $\mathsf{L}\,u = 0$, and the α_j are distinct then $\{e^{\alpha_1 x}, \dots, e^{\alpha_k x}\}$ is a basis of solutions for $\mathsf{L}\,u = 0$.

There are other properties, but the above are all we need.
Thus the differential equation $(**)$ can be rewritten as:

$$(\mathsf{D}^2 - \lambda)u = 0.$$

Now the constant λ maybe zero, positive or negative. We consider thse cases separately:

Case 1: $\lambda = 0$. Then $(**)$ reduces to:

$$u'' = 0$$

and integrating twice we obtain

$$u = A + Bx$$

for some constants A, B. Applying the first initial condition, we have:

$$0 = u(0) = A,$$

leaving us with $u = Bx$, and hence $u' = B$. So applying the second initial condition, we have:

$$2 = u'(0) = B.$$

Hence, $f'(x) = u(x) = 2x$, so that after one more integration we have:

$$f(x) = x^2 + C,$$

for some constant C.

Case 2: $\lambda > 0$. For convenience write $\lambda = \omega^2$. Then $(**)$ becomes

$$\begin{aligned} 0 &= (\mathsf{D}^2 - \omega^2)u \\ &= (\mathsf{D} - \omega)(\mathsf{D} + \omega)u. \end{aligned}$$

Hence $\{e^{\omega x}, e^{-\omega x}\}$ is a basis of solutions.
Equivalently, since each of $\cosh(\omega x)$ and $\sinh(\omega x)$ are linear combinations of $e^{\omega x}$ and $e^{-\omega x}$, i $\{\cosh(\omega x), \sinh(\omega x)\}$ can be taken as the basis of solutions, and hence the general solution of $(**)$ when $\lambda = \omega^2$ is

$$u = A\cosh(\omega x) + B\sinh(\omega x),$$

for some constants A, B. Applying the first initial condition, we have:

$$0 = u(0) = A,$$

leaving us with $u = B\sinh(\omega x)$, and hence $u' = B\omega\cosh(\omega x)$. So applying the second initial condition, we have:

$$2 = u'(0) = B\omega.$$

Hence, $B = 2/\omega$ giving us $f'(x) = u(x) = (2/\omega)\sinh(\omega x)$, so that after one more integration we have:

$$f(x) = (2/\omega^2)\cosh(\omega x) + C,$$

for some constant C, and positive constant ω.

Case 3: $\lambda < 0$. For convenience write $\lambda = -\omega^2$. Then $(**)$ becomes

$$\begin{aligned} 0 &= (\mathsf{D}^2 + \omega^2)u \\ &= (\mathsf{D} - i\omega)(\mathsf{D} + i\omega)u. \end{aligned}$$

Hence $\{e^{i\omega x}, e^{-i\omega x}\}$ is a basis of solutions, or equivalently, $\{\cos(\omega x), \sin(\omega x)\}$ can be taken as the basis of solutions, and hence the general solution of $(**)$ when $\lambda = -\omega^2$ is

$$u = A\cos(\omega x) + B\sin(\omega x),$$

for some constants A, B. Applying the first initial condition, we have:

$$0 = u(0) = A,$$

leaving us with $u = B\sin(\omega x)$, and hence $u' = B\omega\cos(\omega x)$. So applying the second initial condition, we have:

$$2 = u'(0) = B\omega.$$

Hence, $B = 2/\omega$ giving us $f'(x) = u(x) = (2/\omega)\sin(\omega x)$, so that after one more integration we have:

$$f(x) = -(2/\omega^2)\cos(\omega x),$$

for some constant C, and positive constant ω.

Checking, we find all three functions obtained in Cases 1, 2, 3, satisfy the condition $(*)$.

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6. Volume of discussion

Given two real numbers $a, b \in [0, 1]$ such that $a + b = 1$, let D be a closed disc of radius a in $\mathbb{R}^3$ and let D^* be the set of points of $\mathbb{R}^3$ whose distance to D is at most b.

Note. The distance of a point P to D is by definition the minimum of the distances from P to points in D.

For what values of a and b is the maximum volume of D^* attained?

Solution. Let O be the centre of D. Then, whatever the point $P \in D^*$, there is a point $P' \in D$ such that $PP' \leqslant b$ and $OP' \leqslant a$. By the Triangle Inequality, $OP \leqslant a + b = 1$. Therefore, whatever the values of a and b, D^* is contained in the ball B of radius 1 and centre O. The volume of D^* is therefore bounded above by $4\pi/3$, and indeed the bound is attained and $D^* = B$ when $a = 0$ and $b = 1$.

If $a > 0$ and $b < 1$, all the points of D^* contained in the right cylinder having D as a cross-section are at a distance of at most $b < 1$ from the plane containing D, and hence the volume of D^* is less than that of B because B contains two spherical caps of volume greater than 0 disjoint from D^*.

Thus, the maximum volume $4\pi/3$ of D^* is only achieved when $a = 0$ and $b = 1$.

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7. Mind your ps and qs

Determine the pairs (p, q) of positive integers such that

$$1^p + 2^p + \cdots + n^p = (1 + 2 + \cdots + n)^q,$$

for all $n \in \mathbb{N}$.

Solution. Let

$$1^p + 2^p + \cdots + n^p = (1 + 2 + \cdots + n)^q. \qquad (\dagger)$$

If $p = 1$, then trivially $q = 1$. Now suppose $p \geqslant 2$. Then, for $n = 2$, ($\dagger$) becomes

$$1 + 2^p = 3^q,$$

so that $p \geqslant 3$ and, hence, 2^p is a multiple of 8 and so

$$3^q \equiv 1 \pmod 8,$$

whence q is even. Let $q = 2q'$. Then

$$\begin{aligned} 2^p &= 3^{2q'} - 1 \\ &= (3^{q'} - 1)(3^{q'} + 1), \end{aligned}$$

and so we have that $3^{q'} - 1$ and $3^{q'} + 1$ are two powers of 2, that differ by 2; the only powers of 2 that differ by 2 are 2 and 4. So, we have, $3^{q'} - 1 = 2$ and hence $q' = 1$ from which we have $q = 2$ and $p = 3$.
In fact, $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2 = (1 + 2 + \cdots + n)^2$ for all $n \in \mathbb{N}$, as can be shown, by induction.
Thus, the only pairs (p, q) of positive integers such that ($\dagger$) holds for all $n \in \mathbb{N}$, are $(1, 1)$ and $(3, 2)$.

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8. A story well-told

Correspondents $C_1, C_2, \ldots, C_n$ communicate with each other by letters. Each correspondent knows 1 detail of a certain story that has piqued their interest, but the n details of their collective knowledge are all different. Whenever one of the correspondents sends a letter to another correspondent, they tell the other correspondent everything they know of the story, at the time of writing the letter.
What is the minimum number of letters that the n correspondents should send to one another so that each of them gets to know all the details of the story?

Solution. The case $n = 1$ being trivial (no letters needed), we will assume $n > 1$. Each correspondent must send at least one letter, otherwise the details of the information they have will never be known to the other correspondents. We don't change the problem if we assume that two letters are never sent at exactly the same time; we can therefore order the letters chronologically in time. After the first $n - 1$ letters have been sent, there is at least one correspondent who has not sent any. Therefore, at that time, no one other than she knows the details of the information she has. Thus, each of the $n - 1$ other correspondents must receive at least one more letter. Hence, at least $(n - 1) + (n - 1) = 2n - 2$ letters are needed to achieve ubiquity. On the other hand, $2n - 2$ letters are sufficient. Indeed, if each of the correspondents, $C_1, C_2, \ldots, C_{n-1}$ initially send a letter to C_n, who will then know all n details of the story, and then C_n sends each of $C_1, C_2, \ldots, C_{n-1}$ the complete story, all n correspondents know all n details of the story, with $2n - 2$ letters, in all, having been sent.
Thus, for all $n \in \mathbb{N}$, the minimum number of letters necessary to achieve ubiquitous knowledge of n details of a story among n correspondents is $2n - 2$.

9. Row distinction

For which values of the integer n is there an $n \times n$ square matrix, whose entries are either 0 or 1, such that the sums of the entries in the n rows are all different and the sums of the entries in the n columns are all equal?

Solution. Such a square matrix exists for all $n \in \mathbb{N}$. Below we provide one family of examples. For $n = 1$ to 4, the matrices are:

$$\begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}, \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1\end{bmatrix}.$$

In general:

If n is odd, the first row consists of all 1s.
If n is even, the first two rows are respectively all 0s, and all 1s.

For the remaining even number of rows, which we index by m, the m^{th} row consists of

k 1s, followed by $(n-k)$ 0s, if $m = 2k-1$, or
k 0s, followed by $(n-k)$ 1s, if $m = 2k$.

Thus the row sums start: n (for n odd); or $0, n$ (for n even), and then go:

$1, n-1, 2, n-2, \ldots, (n-1)/2, n-(n-1)/2$, if n is odd where $n-(n-1)/2 = (n+1)/2$, or
$1, n-1, 2, n-2, \ldots, n/2-1, n-(n/2-1)$, if n is even, where $n-(n/2-1) = n/2+1$.

For the column sums, except for the first row for n odd, all the rows are in pairs where entries are are 0 or 1 in one partner and their complement 1 or 0 in the other partner. Thus, column sums are:

$(n-1)/2 + 1 = (n+1)/2$, if n is odd, or
$n/2$, if n is even.

So the matrices so defined, have:

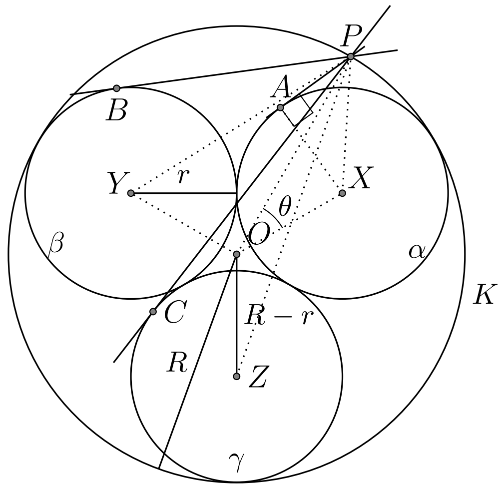
different row sums: $1, 2, \ldots, n$ and equal column sums (to $(n+1)/2$), if n odd, or
row sums $0, 1, \ldots, n/2-1, n/2+1, \ldots, n$ and equal column sums (to $n/2$) if n even.

10. Circular intimacy

In the plane, let α, β, γ be three circles of equal radii, that touch each other pairwise (externally), and are located inside a fourth circle K that touches each of them. From an arbitrary point P on K, tangents are drawn to the circles α, β, γ, to meet these circles at points A, B, C respectively.

For which such points P is one of the distances PA, PB, PC equal to the sum of the other two?

Solution. Let O be the centre of K and let X, Y, Z be the respective centres of α, β, γ. By symmetry, the triangle XYZ is equilateral, and $OX = OY = OZ = R - r$. We claim that for any point P on K, one of the distances PA, PB, PC, is the sum of the two others. Given the symmetry of the problem, it suffices to show that, for P on the $120°$ arc between the points of tangency of α and β with K, $PA + PB = PC$.



Let $\theta = \angle XOP$, and for convenience let $k = R - r$. Then applying the Cosine Rule to triangles XOP, YOP and ZOP we have,

$$PX^2 = R^2 + k^2 - 2kR\cos\theta$$
$$PY^2 = R^2 + k^2 - 2kR\cos(2\pi/3 - \theta)$$
$$PZ^2 = R^2 + k^2 - 2kR\cos(2\pi/3 + \theta)$$

In right triangle PAX,

$$\begin{aligned} PA^2 &= PX^2 - r^2 \\ &= R^2 + k^2 - 2kR\cos\theta - r^2 \\ &= (r+k)^2 + k^2 - 2k(r+k)\cos\theta - r^2 \\ &= 2k^2 + 2kr - 2k(r+k)\cos\theta \\ &= 2k(r+k)(1-\cos\theta) \\ &= 4k(r+k)\sin^2(\theta/2). \end{aligned}$$

Similarly,

$$PB^2 = 4k(r+k)\sin^2(\pi/3 - \theta/2)$$
$$PC^2 = 4k(r+k)\sin^2(\pi/3 + \theta/2)$$

Since $\theta/2$, $\pi/3 - \theta/2$ and $\pi/3 + \theta/2$ all lie in the interval $[0, 2\pi/3]$, their sines are positive, and hence

$$PC - PA - PB = 2\sqrt{k(r+k)}\big(\sin(\pi/3 + \theta/2) - \sin(\theta/2) - \sin(\pi/3 - \theta/2)\big).$$

But, since $\sin(\varphi \pm \psi) = \sin\varphi\cos\psi \pm \cos\varphi\sin\psi$,

$$\sin(\varphi + \psi) - \sin(\varphi - \psi) = 2\cos\varphi\sin\psi.$$

So,

$$\begin{aligned} \sin(\pi/3 + \theta/2) - \sin(\pi/3 - \theta/2) &= 2\cos(\pi/3)\sin(\theta/2) \\ &= 2 \cdot \tfrac{1}{2} \cdot \sin(\theta/2) \\ &= \sin(\theta/2) \end{aligned}$$

Thus,

$$\begin{aligned} PC - PA - PB &= 0 \\ PC &= PA + PB, \end{aligned}$$

which proves our claim.
Thus, one of the distances PA, PB, PC equals the sum of the other two, for any point P on K.