The University of Western Australia
DEPARTMENT OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2021 Problems with Solutions

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers).

1. Truth in inequality

Prove that

$$x^2 + y^2 + 2 \geqslant (x+1)(y+1),$$

for all $x, y \in \mathbb{R}$. When does equality hold?

Solution. Let

$$x^2 + y^2 + 2 \geqslant (x+1)(y+1) \tag{$*$}$$

Then

$$\begin{aligned}
\text{LHS}(*) - \text{RHS}(*) &= x^2 + y^2 + 2 - (x+1)(y+1) \\
&= x^2 + y^2 + 2 - x - y - xy - 1 \\
&= \tfrac{1}{2}(2x^2 + 2y^2 - 2xy - 2x - 2y + 2) \\
&= \tfrac{1}{2}\big(x^2 - 2xy + y^2 + x^2 - 2x + 1 + y^2 - 2y + 1\big) \\
&= \tfrac{1}{2}\big((x-y)^2 + (x-1)^2 + (y-1)^2\big) \\
&\geqslant 0, \quad \text{since a sum of squares is non-negative.} \\
\therefore \text{LHS}(*) &\geqslant \text{RHS}(*), \quad \text{as required.}
\end{aligned}$$

Further, equality holds, if and only if,

$$0 = x - y = x - 1 = y - 1,$$

i.e. if and only if $x = y = 1$.

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2. Naturally cyclic

Find all positive integer quadruples (a, b, c, d) satisfying

$$ab + bc + cd + da = 2021.$$

How many such solutions are there?

Solution. First observe that

$$ab + bc + cd + da = (a+c)(b+d) \text{ and } 2021 = 45^2 - 2^2 = 47 \cdot 43.$$

Also observe that since $a, b, c, d \in \mathbb{N}$, $a+b, c+d \geqslant 2$.
Hence, $(a+c, b+d) = (47, 43)$ or $(43, 47)$, since 43 and 47 are prime. Thus, the set S of natural number quadruples (a, b, c, d) satisfying

$$ab + bc + cd + da = 2021,$$

is:

$$S = \big\{(s, t, 47-s, 43-t), (t, s, 43-t, 47-t) \mid s \in \{1, 2, \dots, 42\}, t \in \{1, 2, \dots, 46\}\big\}$$

and there are $|S| = 2 \cdot 46 \cdot 42 = 2(44^2 - 2^2) = 8 \cdot 483 = 3\,864$ such solutions.

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3. Triply real

Find all real triples (x, y, z) that satisfy the equations

$$x + \frac{1}{x} = 2y^2 \qquad (1)$$

$$y + \frac{1}{y} = 2z^2 \qquad (2)$$

$$z + \frac{1}{z} = 2x^2 \qquad (3)$$

Solution. First observe that $x, y, z \neq 0$, but the RHS of each of (1), (2), (3) is non-negative. Therefore, $x, y, z > 0$, and hence by (3) and the AM-GM Inequality,

$$2x^2 = z + \frac{1}{z} \geqslant 2$$

$$\therefore x \geqslant 1.$$

Similarly, $y, z \geqslant 1$. Hence,

$$2x = x + x \geqslant x + \frac{1}{x} = 2y^2$$

$$\therefore x \geqslant y^2.$$

Similarly,

$$y \geqslant z^2$$

$$z \geqslant x^2$$

Therefore, multiplying corresponding sides of the last 3 inequalities,

$$xyz \geqslant (xyz)^2$$

$$1 \geqslant xyz$$

But $x, y, z \geqslant 1$. So, we have $x = y = z = 1$.
Hence, $(1, 1, 1)$ is the only triple (x, y, z) satisfying (1), (2), (3).

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4. Dinothesaurus

A mad editor has undertaken to publish the list, in alphabetical order, of all the "words" of 26 letters comprising each of the letters of the roman alphabet, exactly once. The gigantic list is to appear in 21 volumes each containing the same number of words. So the first word of the first volume will be

$$abcdefghijklmnopqrstuvwxyz$$

followed by

$$abcdefghijklmnopqrstuvwxzy.$$

What will be the last word of the first volume?

Solution. Let W be the required word. First observe that each volume contains $\frac{1}{21} \cdot 26!$ "words" and that $25! < \frac{1}{21} \cdot 26! < 2 \cdot 25!$. Observe that there are 25! "words" beginning with a, then 25! "words" beginning with b. So the last word of the first volume begins with b. The first "word" beginning with b is:

$$bacdefghijklmnopqrstuvwxyz.$$

Thus, we see that that the next letter is determined by discovering how many times the reduced alphabet $acdefghijklmnopqrstuvwxyz$ of 25 letters is cycled through. The first 24! subwords begin with a, the next 24! begin with c and so on.
Now,

$$\begin{aligned}
\tfrac{1}{21} \cdot 26! &= \tfrac{26}{21} \cdot 25! \\
&= 1 \cdot 25! + \tfrac{5}{21} \cdot 25! \\
&= 1 \cdot 25! + \tfrac{125}{21} \cdot 24! \\
&= 1 \cdot 25! + 5 \cdot 24! + \tfrac{20}{21} \cdot 24! \\
&= 1 \cdot 25! + 5 \cdot 24! + \tfrac{160}{7} \cdot 23! \\
&= 1 \cdot 25! + 5 \cdot 24! + 22 \cdot 23! + \tfrac{6}{7} \cdot 23! \\
&= 1 \cdot 25! + 5 \cdot 24! + 22 \cdot 23! + \tfrac{138}{7} \cdot 22! \\
&= 1 \cdot 25! + 5 \cdot 24! + 22 \cdot 23! + 19 \cdot 22! + \tfrac{5}{7} \cdot 22! \\
&= 1 \cdot 25! + 5 \cdot 24! + 22 \cdot 23! + 19 \cdot 22! + \tfrac{110}{7} \cdot 21! \\
&= 1 \cdot 25! + 5 \cdot 24! + 22 \cdot 23! + 19 \cdot 22! + 15 \cdot 21! + \tfrac{5}{7} \cdot 21! \\
&= 1 \cdot 25! + 5 \cdot 24! + 22 \cdot 23! + 19 \cdot 22! + 15 \cdot 21! + 15 \cdot 20!
\end{aligned}$$

Thus W, the $\frac{1}{21} \cdot 26!^{\text{th}}$ word, is:

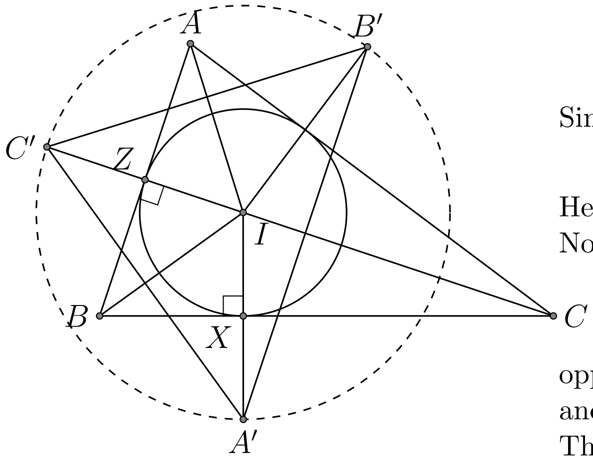
b (the 2^{nd} letter of $abcdefghijklmnopqrstuvwxyz$) followed by,
g (the 6^{th} letter of the reduced alphabet $acdefghijklmnopqrstuvwxyz$) followed by,
y (the 23^{rd} letter of the reduced alphabet $acdefhijklmnopqrstuvwxyz$) followed by,
v (the 20^{th} letter of the reduced alphabet $acdefhijklmnopqrstuvwxz$) followed by,
r (the 16^{th} letter of the reduced alphabet $acdefhijklmnopqrstuwxz$) followed by,
q (the 15^{th} letter of the reduced alphabet $acdefhijklmnopqstuwxz$) followed by,
$zxwutsponmlkjihfedca$ (the last "word" of the reduced alphabet $acdefhijklmnopstuwxz$).

i.e. the last word of the first volume is: $bgyvrqzxwutsponmlkjihfedca$.

5. An incentred triangle

Let triangle ABC have incentre I. Let A', B', C' be the reflections of I in sides BC, CA, AB, respectively.
Prove that $\angle A'B'C'$ does not depend on $\angle BAC$, and find $\angle A'B'C'$ in terms of $\angle ABC$.

Solution. Let r be the inradius of $\triangle ABC$, and let X and Z be the points where the incircle of $\triangle ABC$ touch BC and AB, respectively. Then since A' is the reflection of I in BC,



$$IA' = 2r.$$

Similarly,

$$IC' = IB' = 2r.$$

Hence $I = \text{circumcentre}(A'B'C')$.
Now since,

$$\angle IZB = 90^\circ = \angle IXB,$$

opposite angles of quadrilateral $BZIX$ at Z and X are supplementary.
Therefore, $BZIX$ is cyclic.

So now we have,

$$\begin{aligned}
\angle A'B'C' &= \tfrac{1}{2}\angle A'IC', && \text{(angle at centre is twice angle at circumference on same arc)}\\
&= \tfrac{1}{2}\angle ZIX, && \text{(same angle)}\\
&= \tfrac{1}{2}(180^\circ - \angle ZBX), && \text{(opposite angles in cyclic } BZIX \text{ are supplementary)}\\
&= 90^\circ - \tfrac{1}{2}\angle ZBX\\
&= 90^\circ - \tfrac{1}{2}\angle ABC && \text{(same angle).}
\end{aligned}$$

In particular, $\angle A'B'C'$ depends only on $\angle ABC$ and not on $\angle BAC$.