The University of Western Australia
DEPARTMENT OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2022 Problems with Solutions

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers).

1. Easy but complex

Suppose $z_1, z_2 \in \mathbb{C}$ such that

$$|z_1| = |z_1 + z_2| = 3$$
$$|z_1 - z_2| = 3\sqrt{3}.$$

Determine $\lfloor \log_3 |(z_1\overline{z_2})^{2022} + (\overline{z_1}z_2)^{2022}| \rfloor$, where $\lfloor x \rfloor$ is the *floor* of x (the largest integer less than or equal to $x \in \mathbb{R}$), and $|z|$ and $\overline{z}$ are, respectively, the *modulus* and *conjugate* of $z \in \mathbb{C}$.

Solution. Recall the general properties for $z, z_1, z_2 \in \mathbb{C}$,

$$|z|^2 = z\overline{z} \qquad \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$
$$z + \overline{z} = 2\,\mathcal{R}e\, z \qquad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

Then for the given $z_1, z_2 \in \mathbb{C}$, we have

$$\begin{aligned}
9 = 3^2 &= |z_1 + z_2|^2 \\
&= (z_1 + z_2)(\overline{z_1 + z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} \\
&= |z_1|^2 + 2\,\mathcal{R}e(z_1\overline{z_2}) + |z_2|^2 && (1) \\
27 = (3\sqrt{3})^2 &= |z_1 - z_2|^2 \\
&= |z_1|^2 - 2\,\mathcal{R}e(z_1\overline{z_2}) + |z_2|^2 && (2) \\
\therefore\ \tfrac{1}{2}\big((1) - (2)\big):\ -9 &= 2\,\mathcal{R}e(z_1\overline{z_2}) \\
\tfrac{1}{2}\big((1) + (2)\big):\ 18 &= |z_1|^2 + |z_2|^2 \\
&= 9 + |z_2|^2, \text{ since } |z_1| = 3 \\
\therefore\ |z_1| &= |z_2| = 3 \\
\therefore\ |z_1\overline{z_2}| &= |z_1|\,|\overline{z_2}| = |z_1|\,|z_2| = 9 \\
\therefore\ z_1\overline{z_2} &= 9\operatorname{cis}\theta \text{ for some } \theta \in (-\pi, \pi] \\
\therefore\ -\tfrac{9}{2} = \mathcal{R}e(z_1\overline{z_2}) &= 9\cos\theta \\
-\tfrac{1}{2} &= \cos\theta \\
\pm\tfrac{1}{2}\sqrt{3} &= \sin\theta \\
\therefore\ \{z_1\overline{z_2}, \overline{z_1}z_2\} &= \{9\omega, 9\omega^2\}, \text{ where } \omega \text{ is a complex cube-root of unity} \\
\therefore\ (z_1\overline{z_2})^{2022} + (\overline{z_1}z_2)^{2022} &= 9^{2022}\omega^{2022} + 9^{2022}\omega^{4044} \\
&= 2 \cdot 9^{2022}, \text{ since } 3 \,\big|\, 2022 \implies \omega^{2022} = \omega^{4044} = 1 \\
\therefore\ \big\lfloor \log_3 |(z_1\overline{z_2})^{2022} + (\overline{z_1}z_2)^{2022}| \big\rfloor &= \lfloor \log_3(2) + 4044 \rfloor \\
&= 4044.
\end{aligned}$$

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2. Annually cubic

Let $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha^3 - 3\alpha^2 + 23\alpha &= 20, \text{ and}\\ \beta^3 - 3\beta^2 + 23\beta &= 22.\end{aligned}$$

Find $\alpha + \beta$.

Solution. We have $f(\alpha) = 20$ and $f(\beta) = 22$, where $f(x) = x^3 - 3x^2 + 23x$.
Observe that $f(x-1)$ will have no x^2 coefficient.
Let $u = x - 1$, or equivalently $x = u + 1$:

$$\begin{aligned}f(x) = f(u+1) &= (u+1)^3 - 3(u+1)^2 + 23(u+1)\\ &= u^3 + (3-3)u^2 + (3 - 3\cdot 2 + 23)u + (1 - 3 + 23)\\ &= u^3 + 20u + 21\\ &= (x-1)^3 + 20(x-1) + 21.\end{aligned}$$

Let $g(x) = x^3 + 20x$.
Since $g'(x) = 3x^2 + 20 > 0$ for all $x \in \mathbb{R}$, g is strictly increasing and hence one-to-one.
Therefore, α and β are uniquely determined by

$$\begin{aligned}g(\alpha - 1) &= f(\alpha) - 21 = -1 \text{ and}\\ g(\beta - 1) &= f(\beta) - 21 = 1.\end{aligned}$$

Moreover, g is an odd function. Hence,

$$\begin{aligned}\alpha - 1 &= -(\beta - 1)\\ \therefore\ \alpha + \beta &= 2.\end{aligned}$$

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3. Parabolic sum

For each $n \in \mathbb{N}$, the parabola

$$y = (n^2 + n)x^2 + (2n+1)x + 1,$$

cuts the x-axis at α_n and β_n.

Find $\displaystyle\sum_{n=1}^{2022} \alpha_n \beta_n$.

Solution. Equivalently, α_n and β_n are roots of

$$(n^2 + n)x^2 + (2n+1)x + 1 = 0. \tag{$*$}$$

First we observe that the discriminant of $(*)$,

$$\begin{aligned}\Delta &= (2n+1)^2 - 4(n^2 + n)\\ &= 1 > 0\end{aligned}$$

By Viète's Theorem[1],

$$\alpha_n \beta_n = \frac{1}{n^2 + n}.$$

[1] **Viète's Theorem** states: for a monic quadratic $x^2 + bx + c = 0$ with roots α, β, $b = -(\alpha + \beta)$ and $c = \alpha\beta$.

So we are required to find

$$\begin{aligned}\sum_{n=1}^{2022} \alpha_n\beta_n &= \sum_{n=1}^{2022} \frac{1}{n^2+n}\\ &= \sum_{n=1}^{2022} \frac{1}{n(n+1)}\\ &= \sum_{n=1}^{2022} \frac{1}{n} - \frac{1}{n+1}\\ &= 1 - \frac{1}{2} + \frac{1}{2} - \cdots - \frac{1}{2022} + \frac{1}{2022} - \frac{1}{2023}\\ &= \frac{2022}{2023}.\end{aligned}$$

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4. Alternating sums

Let the decimal digit representation of $N \in \mathbb{N}$ be $\overline{a_n a_{n-1} \dots a_1 a_0}$, i.e.

$$N = a_0 + 10a_1 + 10^2 a_2 + \cdots + 10^n a_n.$$

and let $A(N)$ be the *alternating sum of digits* of N. i.e.

$$A(N) = a_0 - a_1 + a_2 - \cdots + (-1)^n a_n.$$

Also, we define $A(0) = 0$, and for $N < 0$, $A(N) = -A(-N)$ (what one gets by thinking of all the digits of N as negative).

Suppose $\alpha = 2022^{2022}$, $A(\alpha) = \beta$, and $A(\beta) = \gamma$.

Knowing that $A(\gamma)$ is negative, what is $A(\gamma)$?

Solution. First observe that since $\alpha = 2022^{2022} < (10^4)^{2022}$, α has at most 8088 digits. Hence

$$\begin{aligned}-36396 = A(\underbrace{9090\dots9090}_{4044 \text{ copies of } 90}) \leqslant A(\alpha) &\leqslant A(\underbrace{0909\dots0909}_{4044 \text{ copies of } 09}) = 36396\\ -21 = A(-30909) \leqslant A(\beta) &\leqslant A(30909) = 21\\ -9 = A(-09) \leqslant A(\gamma) &\leqslant A(09) = 9\end{aligned}$$

Now, since $N \equiv A(N) \pmod{11}$,

$$\begin{aligned}\alpha &\equiv A(\alpha) \pmod{11}\\ &\equiv A(\beta) \pmod{11}\\ &\equiv A(\gamma) \pmod{11}\end{aligned}$$

but

$$\begin{aligned}\alpha &= 2022^{2022}\\ &\equiv \big(A(2022)\big)^{2022} \pmod{11}\\ &\equiv \big((-2)^{10}\big)^{202} \cdot (-2)^2 \pmod{11}\\ &\equiv 1 \cdot 4 \pmod{11}, \text{ by Fermat's Little Theorem.}\end{aligned}$$

Therefore, $A(\gamma) = 4$ or -7, so that knowing that $A(\gamma)$ is negative, we have $A(\gamma) = -7$.

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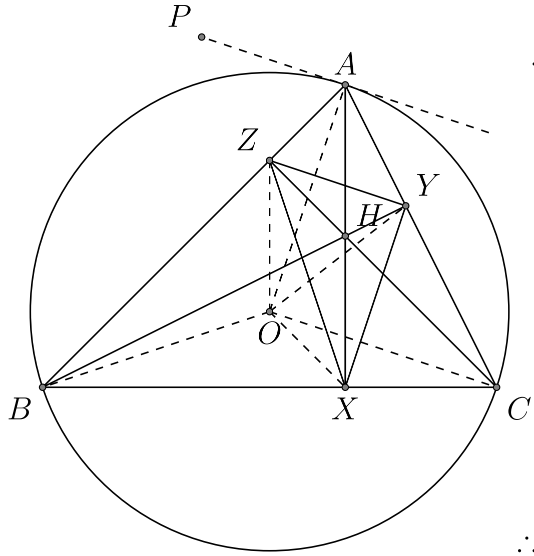
5. Perimetric area

The diameter of the circumcircle of acute triangle ABC is 4, and its area is S, with X, Y, Z points on sides BC, CA, AB respectively.
Prove that AX, BY, CZ are altitudes of $\triangle ABC$ if and only if $S = ZY + YX + XZ$.

Solution. Let $K = \text{circumcircle}(ABC)$, $O = \text{circumcentre}(ABC)$, and let P be a point on the tangent to K through A.
($\Longrightarrow$) Assume AX, BY, CZ are altitudes of $\triangle ABC$. Then,

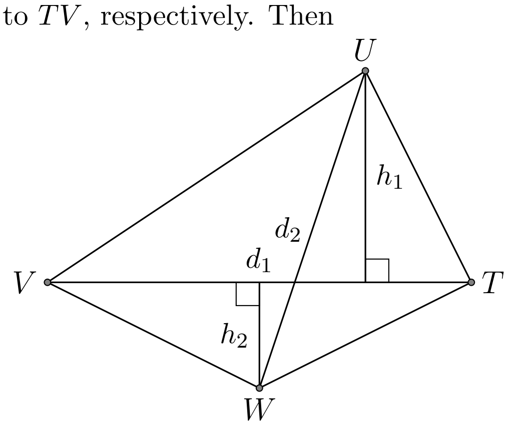


$$
\begin{aligned}
\angle BZC &= 90^\circ \\
&= \angle BYC \\
\therefore\ BZYC &\text{ is cyclic,} && \text{(equal angles on common chord } BC) \\
\therefore\ \angle AZY &= \angle YCB, && \text{(exterior angle of cyclic } BZYC \text{ equals interior opposite angle)} \\
&= \angle ACB, && \text{(same angle)} \\
&= \angle PAB, && \text{(by Tangent-Chord Thm: tangent } PA\text{, chord } AB \text{ to circle } K) \\
&= \angle PAZ, && \text{(same angle)} \\
\therefore\ ZY &\parallel PA, && \text{(alternate angles)} \\
&\perp AO, && (AO \text{ is a radius of } K) \\
\therefore\ |OZAY| &= \tfrac{1}{2} AO \cdot ZY \\
&= ZY, && \text{since } 2\,AO = \text{diam}(K) = 4
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|OYCX| &= YX, \text{ and} \\
|OXBZ| &= XZ \\
\therefore\ S &= |OZAY| + |OYCX| + |OXBZ| \\
&= ZY + YX + XZ.
\end{aligned}
$$

For the converse direction of the proof, we need the following lemma.
Lemma. For quadrilateral $TUVW$, with diagonals d_1, d_2, $|TUVW| \leqslant \frac{1}{2} d_1 d_2$.
Moreover, equality occurs if and only if $TV \perp UW$.
Proof. Let $d_1 = TV$, $d_2 = UW$. Drop altitudes h_1, h_2 of triangles TUV, TWV from U, W to TV, respectively. Then



$$
\begin{aligned}
|TUVW| &= |TUV| + |TWV| \\
&= \tfrac{1}{2} d_1 h_1 + \tfrac{1}{2} d_1 h_2 \\
&= \tfrac{1}{2} d_1 (h_1 + h_2) \\
&\leqslant \tfrac{1}{2} d_1 d_2.
\end{aligned}
$$

Moreover, for equality to occur altitudes h_1, h_2 must coincide with diagonal d_2, which occurs if and only if $TV \perp UW$.

($\Longleftarrow$) Now we assume $S = ZY + YX + XZ$. Then

$$
\begin{aligned}
|OZAY| &\leqslant \tfrac{1}{2} OA \cdot ZY, && \text{(by Lemma above)} \\
&= ZY && (1) \\
|OYCX| &\leqslant YX && (2) \\
|OXBZ| &\leqslant XZ && (3)
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ ZY + YX + XZ &= S \\
&= |OZAY| + |OYCX| + |OXBZ| \\
&\leqslant ZY + YX + ZZ.
\end{aligned}
$$

$\therefore$ $(1), (2), (3)$ are equalities

$$
\begin{aligned}
\therefore\ ZY &\perp OA, && \text{(by Lemma above)} \\
OA &\perp PA \\
\therefore\ ZY &\parallel PA
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \angle AZY &= \angle PAZ, && \text{(alternate angles)} \\
&= \angle PAB, && \text{(same angle)} \\
&= \angle ACB, && \text{by Tangent-Chord Thm} \\
&= \angle YCB, && \text{(same angle)}
\end{aligned}
$$

$\therefore$ $BZYC$ is cyclic

Similarly, $AYXB, CXZA$ are cyclic

$$
\begin{aligned}
\therefore\ \angle AYB &= 180^\circ - (\angle ABY + \angle BAY), && \text{(angles of } \triangle ABY\text{)} \\
&= 180^\circ - (\angle ZBY + \angle CAZ), && \text{(same angles)} \\
&= 180^\circ - (\angle ZCY + \angle CAZ), && \text{(angles on chord } ZY \text{ of cyclic } BZYC\text{)} \\
&= 180^\circ - (\angle ZCA + \angle CAZ), && \text{(same angle)} \\
&= \angle AZC, && \text{(angles of } \triangle ZCA\text{)} \\
\angle AYB + \angle AZC &= \angle AXB + \angle AXC, && \text{(angles on chords } AB, AC \text{ in cyclic } AYXB, CXZA\text{, respectively)} \\
&= 180^\circ, && \text{(straight angle)}
\end{aligned}
$$

$\therefore$ $\angle AYB = \angle AZC = 90^\circ$

$\therefore$ BY, CZ are altitudes of $\triangle ABC$

Similarly, AX is an altitude of $\triangle ABC$.