The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

1997 Problems

1. L and M are lines in $\mathbb{R}^3$ such that L lies in a plane perpendicular to M. Show that M lies in a plane perpendicular to L.

---

2. A *partition* of a set S is a collection of disjoint subsets whose union is S. For a partition π of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π' of S, there are two distinct numbers x and y such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

---

3. Find the equation of the line tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

in the first quadrant that forms with the coordinate axes the triangle of smallest possible area.

---

4. Let f be a continuous real function on $\mathbb{R}$ and let I be the identity map on $\mathbb{R}$. Show that:

(a) If $f \circ f = I$, then $f = I$ or f is decreasing.

(b) If $f \circ f \circ f = I$ then $f = I$.

(c) If $f \circ f \circ f \circ f = I$ then $f \circ f = I$.

---

5. Elastic thread is to be wound on to a cylindrical bobbin which is 2 cm in diameter and 5 cm long, so that the curved surface is covered by one thickness of thread. The thread is originally 15 m long and 0.02 cm in diameter with a circular cross-section, but it needs to be stretched in order to do this job. When stretched, the thickness of the thread decreases but its cross-section remains circular and its total volume remains constant.
Find the length of the stretched thread.

∗6. (a) Positive integers $n_1, n_2, \ldots$ (not necessarily distinct) are to be chosen so that their sum is 100. What is the maximum value of their product?

(b) What happens if the word 'integers' is replaced by 'real numbers'?

∗7. Show that every square real matrix is the product of two real symmetric matrices.

∗8. Show that all solutions of the differential equation $y'' + e^x y = 0$ are bounded as $x \to \infty$.

∗9. Let $\mathcal{F}$ be the set of all intervals, closed, open and half-open, in $\mathbb{R}$ and let $\mathcal{G}$ be the set of all closed intervals in $\mathbb{R}$. Let f be a real function defined on $\mathbb{R}$. Show that f is continuous if and only if for every $F \in \mathcal{F},\ f(F) \in \mathcal{F}$ and for every $G \in \mathcal{G},\ f(G) \in \mathcal{G}$.
Generalise this result to $\mathbb{R}^n$.

∗10. Let S be a set, and for all $x,\ y \in S$, let $x * y \in S$ such that

(a) $x * (x * y) = y$ and

(b) $(y * x) * x = y$.

Show that $*$ is commutative but not necessarily associative.
