The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

1998 Problems with Solutions

1. **A coin-tossing game.** Alice tosses 39 fair coins and Bob tosses 40. What is the probability that Bob gets more heads than Alice?

   Solution. [**By Ng Beng Siong, 3rd Year, UWA**]
   Since Bob tosses once more than Alice, either Bob tosses more heads than Alice or Bob tosses more tails than Alice. Since the coins are fair, these two events are equally likely. Hence the probability that Bob tosses more heads than Alice is 0.5.

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2. **Chasing bugs.** Four bugs are situated at the vertices of a square of side 1 cm and begin chasing each other, each bug running at the same constant speed 1 cm/s directly towards the next bug in the anti-clockwise direction. What distance is travelled by each bug before mutual capture?

   Solution. By symmetry, the bugs are always at the vertices of a square of decreasing size. Hence B_2 is always moving in a direction perpendicular to B_1. So the speed of closure of these two is always 1 cm/s. Since they start 1 cm apart, they meet after 1 second, having travelled 1 cm.

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3. **Lines and Planes.** Let AB and CD be non-coplanar lines in $\mathbb{R}^3$. Let π be a plane cutting AB at P, BC at Q, CD at R and DA at S.
   Prove that
$$\frac{|AP|}{|PB|}\frac{|BQ|}{|QC|}\frac{|CR|}{|RD|}\frac{|DS|}{|SA|} = 1.$$

   Solution. [**By Tim Hillman**]
   With respect to the independent vectors $\overrightarrow{PQ}$, $\overrightarrow{PS}$ and $\overrightarrow{PA}$, $P = (0,0,0)$, $Q = (1,0,0)$, $S = (0,1,0)$, $A = (0,0,1)$ and $R = (\ell, m, 0)$ for some scalars ℓ and m.
   Thus if $\overrightarrow{AP} = a\,\overrightarrow{AB}$, $\overrightarrow{BQ} = b\,\overrightarrow{BC}$, $\overrightarrow{CR} = c\,\overrightarrow{CD}$ and $\overrightarrow{DS} = d\,\overrightarrow{DA}$ then
$$\begin{aligned} B &= \Big(0,0,\frac{a-1}{a}\Big),\\ C &= \Big(\frac{1}{b},0,\frac{(a-1)(b-1)}{ab}\Big) \text{ and}\\ D &= \Big(\frac{\ell b+c-1}{bc},\frac{m}{c},\frac{(a-1)(b-1)(c-1)}{abc}\Big). \end{aligned}$$
   Also
$$\begin{aligned} \overrightarrow{PA} &= \overrightarrow{PS} + \overrightarrow{SA}\\ &= \frac{1}{d}S + \frac{d-1}{d}D\\ \therefore (0,0,1) = A &= \Big(\ldots,\ldots,\frac{(a-1)(b-1)(c-1)(d-1)}{abcd}\Big)\\ \therefore 1 &= \frac{(a-1)(b-1)(c-1)(d-1)}{abcd} \end{aligned}$$

Now,

$$\frac{|AP|}{|PB|}\frac{|BQ|}{|QC|}\frac{|CR|}{|RD|}\frac{|DS|}{|SA|} = \left|\frac{abcd}{(a-1)(b-1)(c-1)(d-1)}\right| = 1$$

as required.

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4. Minimum fold. Given a rectangular sheet of paper, how do you fold the bottom right corner to the left side so that the folded portion has minimum area?

Solution. Label the sheet $ABCD$ with short edges AB and CD. Suppose the dimensions are $a \times b$ with width $a <$ height b. (The problem doesn't really make sense if $a > b$.) Fold the sheet at the point E of the short edge AB so that the corner B meets the long edge AD at the point B'. Let $|EB| = |EB'| = x$ so $|AE| = a - x$ and $|BC| = |B'C| = b$. Let $\angle AB'C = \angle B'CD = \theta$ so $\sin\theta = (a-x)/x$ and $\cos\theta = a/b$.

The area of the fold $A = \frac{1}{2}bx = \frac{1}{2}(a/\cos\theta) = a^2/\big(2\cos\theta + \sin(2\theta)\big)$.

Considering A as a function of θ,

$$\frac{dA}{d\theta} = a^2\frac{2\sin\theta - 2\cos(2\theta)}{\big(2\cos\theta + \sin(2\theta)\big)^2}, \quad \theta \in (0, \pi/2).$$

The denominator is positive, so $dA/d\theta = 0$ if and only if $\sin\theta = \cos 2\theta = 1 - 2\sin^2\theta$ or $(2\sin\theta - 1)(\sin\theta + 1) = 0$.

Only the solution $2\sin\theta = 1$ satisfies $\theta \in (0, \pi/2)$, so $dA/d\theta = 0$ if and only if $\theta = \pi/6$.

To check for minimality, note that $2\sin\theta - 2\cos 2\theta < 0$ for $\theta \in (0, \pi/6)$ and > 0 for $\theta \in (\pi/6, \pi/2)$.

Hence the fold has minimum area for $x = a/\big(1 + \sin(\pi/6)\big) = 2a/3$.

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5. Let $N(x)$ denote the number of positive integers less than x that are not divisible by 2 or 3. Show that

$$\lim_{x\to\infty}\frac{N(x)}{x} = \frac{1}{3}.$$

Solution. Between every pair of multiples of 6, there are 2 numbers that are coprime to 6. Given x, let y be the largest multiple of $6 \le x$ so the number of numbers coprime to 6 $\le y/3$. Since $x < y + 6$, there are at most 2 numbers coprime to 6 between x and y. Hence $y/3 \le N(x) \le y/3 + 2$.

Since $x - 6 \le y < x$, $(x-6)/3 \le N(x) \le x/3 + 2$. Consequently,

$$\frac{1}{3} = \lim_{x\to\infty}\frac{x-6}{3x} \le \lim_{x\to\infty}\frac{N(x)}{x} \le \lim_{x\to\infty}\frac{x/3+2}{x} = \frac{1}{3}.$$

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6. From a point P on an ellipse a chord is drawn perpendicular to the tangent at P meeting the ellipse again at Q. Prove or disprove that the shortest such chord is the minor axis.

Solution. The shortest chord length is $(\sqrt{27}a^2b^2)/(a^2+b^2)^{3/2}$, so the minor axis is the shortest if and only if $(\sqrt{27}a^2b^2)/(a^2+b^2)^{3/2} < 2b$.

To prove this, let the ellipse have cartesian equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, \quad x \in (-a,a)$.

By implicit differentiation the slope at any point $P(x_0,y_0)$ on the ellipse is $\dfrac{dy}{dx}=\dfrac{-x_0b^2}{y_0a^2}$.

Hence the equation of the normal through P is

$$y=\Big(\frac{a}{bx_0}\sqrt{a^2-x_0^2}\Big)x+\Big(\frac{b}{a}-\frac{a}{b}\Big)\sqrt{a^2-x_0^2}.$$

This normal meets the ellipse again at $Q(x_1,y_1)$ where

$$y_1=\frac{b}{a}\sqrt{a^2-x_1^2}=\Big(\frac{a}{bx_0}\sqrt{a^2-x_0^2}\Big)x_1+\Big(\frac{b}{a}-\frac{a}{b}\Big)\sqrt{a^2-x_0^2}.$$

Squaring, we get

$$\frac{b^2}{a^2}(a^2-x_1^2)=\frac{a^2}{b^2x_0^2}(a^2-x_0^2)x_1^2+\Big(\frac{b}{a}-\frac{a}{b}\Big)^2(a^2-x_0^2)+\frac{2a}{bx_0}\Big(\frac{b}{a}-\frac{a}{b}\Big)(a^2-x_0^2)x_1.$$

This is a quadratic equation for x_1. Solving it by the quadratic formula is horrendous, but remember that x_0 is a root, so we can find the other root by dividing by x_1-x_0.

The result simplifies to

$$x_1=x_0\frac{-a^6+2a^4b^2+a^4x_0^2+b^4x_0^2-2a^2b^2x_0^2}{a^6-a^4x_0^2+b^4x_0^2}.$$

It remains to calculate $y_1=(b/a)\sqrt{a^2-x_1^2}$ and $d^2=(x_1-x_0)^2+(y_1-y_0)^2$ in terms of x_0, a and b and differentiate it with respect to x_0. The result is the value of x_0 which minimises the chord length, given at the beginning of the solution.

An alternative way to attack this problem (one of the Japanese Temple Problems discussed in the May, 1998 Scientific American, is to find a specific ellipse for which the shortest chord is the minor axis, e.g. a circle, and another for which the shortest chord is definitely not the minor axis. For example, **Damian Bramanis, First Year, UWA** took $x^2/10^{10}+y^2=1$ and found a chord PQ as above of length < 2.

∗7. Imperfect mirrors. Suppose a type of glass is such that:

(i) 70% of light shining from one side is transmitted through to the other.

(ii) 20% of the light is reflected.

(iii) The remaining 10% is absorbed in the glass.

How much of an original light source perpendicular to the glass will be transmitted through 3 parallel panes of glass?

Solution. [**By Tim Hillman, 3rd Year, UWA**]
Suppose the light source is on the left. Let p be the fraction of a ray of light entering pane 1 from the left which is eventually transmitted from pane 3 to the right.

Let q be the fraction entering pane 1 from the right which is eventually transmitted from pane 3 to the right.

Let r be the fraction entering pane 2 from the left which is eventually transmitted from pane 3 to the right.

Let s be the fraction entering pane 2 from the right which is eventually transmitted from pane 3 to the right.

Let t be the fraction entering pane 3 from the left which is eventually transmitted from pane 3 to the right.

Now p, r, s and t are related by the linear equations $p = \frac{7}{10}r$, $q = \frac{1}{5}r$, $r = \frac{7}{10}t + \frac{1}{5}q$, $s = \frac{7}{10}q + \frac{1}{5}t$ and $t = \frac{7}{10} + \frac{1}{5}s$. This linear system has augmented matrix

$$\begin{bmatrix} 1 & 0 & -7/10 & 0 & 0 & | & 0 \\ 0 & 1 & -1/5 & 0 & 0 & | & 0 \\ 0 & -1/5 & 1 & 0 & -7/10 & | & 0 \\ 0 & -7/10 & 0 & 1 & -1/5 & | & 0 \\ 0 & 0 & 0 & -1/5 & 1 & | & 7/10 \end{bmatrix}$$

with solution $p = 343/902,\ q = 49/451,\ r = 245/451,\ s = 203/902,\ t = 336/451.$

The fraction of the original light transmitted through the three panes is $p = 343/902 \approx 0.38027$.

∗8. Passing Pennies. Players $1, 2, 3, \ldots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

Solution. [**From 1997 Putnam Competition**]
Let's assume there is more than one player, so the game can start. We show that the game terminates with one player holding all of the pennies if and only if $n = 2^m + 1$ or $n = 2^m + 2$ for some m. First suppose we are in the following situation for some $k \geq 2$. (Note: for us, a "move" consists of two turns, starting with a one-penny pass.)
Except for the player to move, each player has k pennies;

The player to move has at least k pennies.

We claim then that the game terminates if and only if the number of players is a power of 2. First suppose the number of players is even; then after m complete rounds, every other player, starting with the player who moved first, will have m more pennies than initially, and the others will all have 0. Thus we are reduced to the situation with half as many players; by this process, we eventually reduce to the case where the number of players is odd. However, if there is more than one player, after two complete rounds everyone has as many pennies as they did before (here we need $m \geq 2$), so the game fails to terminate. This verifies the claim. Returning to the original game, note that after one complete round, $\lfloor \frac{n-1}{2} \rfloor$ players remain, each with 2 pennies except for the player to move, who has either 3 or 4 pennies. Thus by the above argument, the game terminates if and only if $\lfloor \frac{n-1}{2} \rfloor$ is a power of 2, that is, if and only if $n = 2^m + 1$ or $n = 2^m + 2$ for some m.

$*$9. The Lemming Problem. Given a 1000×1000 grid of squares, each square containing an arrow pointing in one of the 8 main directions (N, NE, E, etc.) with the following continuity condition: If two squares share an edge then the arrows in them differ by at most 45 degrees (therefore they are the same or they differ by 45 degrees to one side or the other).
Prove that if a lemming starts in any square and then uses the arrows to trace out a path on the grid, he will eventually fall off an edge.

Solution. Suppose the lemming does not fall off an edge. Since there are only a finite number of squares she must eventually revisit a square she has previously been in and since there is exactly one arrow in each square, she must pass through it in the same direction. Hence her path contains a loop.

We want to show that there cannot be any loops on any valid arrangement of arrows. Suppose there exists a valid arrangement of arrows with a loop C. Assume without loss of generality that C goes clockwise. Take out a red magic marker and draw the loop on the board. Now rotate each of the arrows on the board $\pi/4$ clockwise. This produces a new valid arrangement of arrows, and it has the property that if the lemming is placed inside the area outlined in red, it will never get outside. Thus there must be a strictly "smaller" loop D with the new arrangement of arrows, i.e. one with fewer enclosed squares. This is a contradiction, since arbitrarily small loops can't exist in a valid arrangement of arrows.

$*$10. The lonely runner. Four runners set off at constant different speeds to run laps round a 1 km circular track. Prove that each runner, is at some time $\geq 1/4$ km from all the other runners. (The general case has k runners, each at some time $\geq 1/k$ from all the others).

Solution. [**By Damian Bramanis** (slightly modified)]
Let A be any runner. If we take a camera looking down on the track and rotate it at the speed of A, the problem will not be changed, so we can assume that A is stationary. If any of the runners are now running backwards, we can reverse their directions without changing the problem. So we have three runners B the slowest, next C, and D the fastest, and we need to show that at some time they are all in the semicircle L opposite A.

Now C overtakes B at regular periodic intervals. Scale the time factor so that these intervals are one time unit. Call the points of overtaking $E_t : t = 1, 2, \ldots$ and let D_t be the position of D at time t.

Case 1: $E_t \notin L$ for all t. Then speed $C = 2 \times$ speed B. For each period that B is in L, C is in L for half this time and C enters and leaves L once. But speed $D >$ speed C, so at some time t each of B, C and D is in L.

Case 2: $E_t \in L,\ D_t \notin L$. Let α be the bearing of D_t with respect to A and θ the bearing of E_t with respect to A. If $\alpha \geq \theta - \pi$, then D will enter L before B and C leave. If $\alpha < \theta - \pi$ then turn back the clock less than one unit so that B and C are still in L but D has not yet left L.

Case 3: There is some t at which E_t and D_t are both in L. Then $B,\ C$ and D are all in L.