The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2000 Problems with Solutions

1. Product of Rotations. R is anti-clockwise rotation in space by $\pi/2$ about the z-axis, (looking from the positive end towards the origin) and S is anti-clockwise rotation by $\pi/2$ about the y-axis, (looking from the positive end towards the origin). Prove that the composite $R \circ S$ is also a rotation and find the axis and the angle of rotation.

Solution. [**By Jesse Petersen, 2nd year, UWA**]
The standard matrices of the given rotations are:

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Hence the product $R \circ S$ has standard matrix

$$R \circ S = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

One can verify that $R \circ S$ has one eigenvalue $= 1$ with eigenspace the span of $(-1, 1, 1)^\top$. Hence it represents rotation about the axis through $(-1, 1, 1)^\top$ and one can check that the angle of rotation is $120°$.

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2. Matrix addresses. Show that in an $n \times n$ matrix there are $\binom{n}{2}^2$ pairs of entries with one lying above and to the right of the other.

Solution. There are $\binom{n}{2}$ ways to choose a pair of distinct row indices and $\binom{n}{2}$ ways to choose a pair of distinct column indices. So there are $\binom{n}{2}^2$ ways to choose (x_1, x_2, y_1, x_2). Exactly one of the 4 pairs $\big((x_i, y_j), (x_k, y_m)\big)$ has one entry lying above and to the right of the other.

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3. Locus in space. S is a diagonal of the face of a cube and T is that diagonal of the opposite face which is not parallel to S. What is the set of mid-points of all line segments XY with X on the line segment S and Y on the line segment T?

Solution. One can see either geometrically or by analytic geometry that the square whose edges are the line segments joing the midpoints of the four faces not containing S or T is the set of such midpoints.

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4. Crazy Mathematicians.

Alice: I am insane.
Bob: I am pure.
Charlie: I am applied.
Dorothy: I am sane.
Alice: Charlie is pure.
Bob: Dorothy is insane.
Charlie: Bob is applied.
Dorothy: Charlie is sane.

Describe the four mathematicians, given that:

(i) pure mathematicians tell the truth about their beliefs,

(ii) applied mathematicians lie about their beliefs,

(iii) sane mathematicians' beliefs are correct, and

(iv) insane mathematicians' beliefs are incorrect.

Solution. [**By Jesse Petersen, 2nd year UWA**]
The unique solution is:
Alice is an insane applied mathematician,
Bob is a sane pure mathematician,
Charlie is an insane pure mathematician, and
Dorothy is also an insane pure mathematician.

To see this, note that all mathematicians think themselves sane. Hence Dorothy is pure so insane because of what she says about Charlie.
So Bob is pure and insane because of what he says about Dorothy.
So Charlie is pure and insane because of what he says about Bob.
And Alice is applied and insane because of what she says about herself and Charlie.

***5. A calculus problem.** Let f be differentiable on $\mathbb{R}$ and satisfy $|f'(x)| \leq \theta < 1$ for some $\theta \in \mathbb{R}$. Prove that $f(x) = x$ has exactly one solution. What happens if the '$\leq \theta$' is omitted?

Solution. Let $g(x) = f(x) - x$. Then $-\theta - 1 \leq g'(x) \leq \theta - 1$ for all x.
If $g(0) > 0$, let $a = g\big(g(0)/(1-\theta)\big)$. By the Mean Value Theorem there exists c between 0 and a such that

$$\frac{g(a) - g(0)}{a} = g'(c) \leq \theta - 1$$
$$\therefore g(a) \leq g(0) - g(0) = 0.$$

By the Intermediate Value Theorem, there exists x such that $g(x) = 0$.
Similarly, if $g(0) < 0$ there exists x such that $g(x) = 0$.

If '$\leq \theta$' is omitted, consider $f(x) = x - \arctan(x) + \pi/2$. Then $|f'(x)| = x^2/(1+x^2) < 1$ for all x, but $f(x) = x$ would imply the contradiction $\arctan(x) = \pi/2$.

∗6. Loop on a sphere. A flexible and inextensible loop of cable of mass M rests on the surface of a smooth sphere of radius R, the cable forming a horizontal ring of radius r. Find the tension in the cable. What happens as $r \to R$?

Solution. Consider a small element of the chain subtending an angle $\Delta\theta$ at the centre of the circle of radius r.

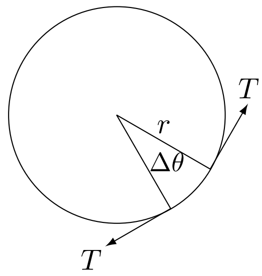

Figure 1(a).

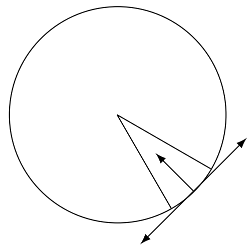

Figure 1(b).

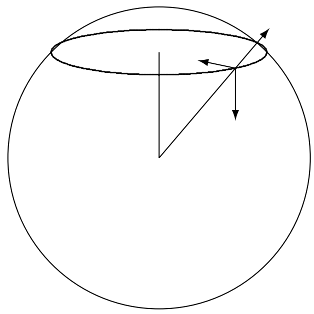

Figure 2.

The tension forces on this element are shown in Figure 1(a), and resolved in Figure 1(b). The resolved component perpendicular to the radius is 0, while the resolved component towards the centre is

$$2T\sin(\tfrac{1}{2}\Delta\theta) \approx 2T \cdot \tfrac{1}{2}\Delta\theta.$$

The forces on the element (see Figure 2) are the weight $\big(\Delta\theta/(2\pi)\big)Mg$ vertically down, the reaction $N\Delta\theta$ normal to the sphere and $T\Delta\theta$ horizontally. Resolving in the direction perpendicular to $N\Delta\theta$ yields

$$T\Delta\theta\cos\phi = \left(\frac{\Delta\theta}{2\pi}\right)Mg\sin\phi,$$

where $\sin\phi = r/R$, $\tan\phi = r/\sqrt{R^2 - r^2}$. Hence

$$T = \frac{Mg}{2\pi}\frac{r}{\sqrt{R^2 - r^2}} \to \infty \text{ as } r \to R.$$

∗7. Ellipse in a paper cup. A paper cup in the shape of a cone contains some water. Show that if you tip the cup without spilling any water, the surface forms an ellipse whose minor axis has length independent of the tipping angle.

Solution. Let the cone be $z^2 = c^2(x^2 + y^2)$ and suppose the initial water level is h. Then the surface is a circle of radius $b = h/c$ and the volume of water is $V = (\pi/3)b^2h = (\pi/3)bA$, where A is the area of the wet triangle in the yz-plane.
When the cup is tipped the surface is an ellipse with minor semi-axis b' say and the volume is V'. So we have to show that $V' = V$ implies $b' = b$, or equivalently that $b' = b$ implies $V = V'$.
So consider cross-sections of the cone by planes parallel to the x-axis producing ellipses with minor semi-axis b. The end-points of the minor semi-axes lie in the planes $x = \pm b$ and their projections in the y-z plane form the hyperbola $H : z^2 = c^2(b^2 + y^2)$, the asymptotes being the lines of intersection of the cone with the y-z plane. The major axis of the boundary ellipse lies in the y-z plane with its end-points on the asymptotes and its mid-point on H.

But all such segments are tangent to H and the triangles formed by the segments tangent to a hyperbola at their midpoints with their end-points on the asymptotes all have the same area: this property is true for the standard hyperbola $xy = 1$ and is invariant under affine transformations.
Now let h' be the height of the tipped cone whose base is the ellipse and vertex is 0 with major semi-axis a. Then the area of the wet triangle is $ah' = A = bh$ so the volume of the tipped cone is $(\pi/3)bah' = (\pi/3)bA = V$.

∗8. Singular matrices. Let A be an $n \times n$ matrix with 0s on the main diagonal and ± 1 off the main diagonal. Show that if n is even, then A is non-singular (invertible), while if n is odd, A may be singular or non-singular.

Solution. Suppose n is even. and look at $AA^\top$. It has all odd numbers down the main diagonal and even numbers off it, so it is non-singular since it is non singular mod 2. hence A itself is non-singular.
If n is odd, and A is skew symmetric it is singular, but if it has all 1s off the main diagonal it is non-singular.

∗9. Mutually perpendicular line segments. From a point in space not on the x-y plane three mutually perpendicular line segments are drawn to the x-y plane.
Show that if a, b and c are the lengths of these line segments then the sum

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

is independent of the orientations of the line segments.

Solution. [**By Desheng Ji, 1st year, Curtin**]
Translate the point to the origin and rotate about a line so that the three mutually orthogonal lines lie along the co-ordinate axes. Thus assume the point is the origin O, the plane is arbitrary, and the line segments are on the the 3 axes, i.e. the plane intersects the x-axis at $C(c, 0, 0)$, the y-axis at $B(0, b, 0)$ and the z-axis at $A(0, 0, a)$.
Then the distance from the origin to the plane is

$$\sqrt{\frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}.$$

Hence

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

is independent of the orientations of the given line segments.

∗10. Measuring sticks. Two rods of length 1 metre are divided into m and n equal parts, where m and n are coprime integers. They are placed side by side so that their ends coincide. Find the minimum distance apart of the division marks and find how many pairs of such minimum distances exist.
If $m = 250$ and $n = 243$, which pairs are at the minimum distance?

Solution. The division points on the first rod are at $i/m, i = 0, 1, \ldots, m-1$; the division points on the second rod are at $j/m, j = 0, 1, \ldots, n-1$. Hence the differences are equal to

$$\left|\frac{i}{m} - \frac{j}{n}\right| = \frac{|ni - mj|}{mn}.$$

So the least difference occurs at the least value of $|ni - mj|$, which is 1, the gcd of m and n. Thus the least difference is $1/(mn)$, and it is attained twice, for if $|ni - mj| = 1$ with $i < m$ and $j < n$, then also $|n(i-m) - m(j-n)| = 1$, but no other combination $|nr - ms| = 1$ if $r < m$ and $s < n$.

If $m = 250$ and $n = 243$, then the minimum distance is $1/60\,750$ attained when $i = 107$ and $j = 104$, by the Euclidean algorithm, and when $i = -143$ and $j = -139$.