The Blakers Mathematics Contest

2004 Questions and Solutions

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1 A unique surd

Show that there is a unique positive solution to

$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + x}}}}$$

Solution: The equation $x = \sqrt{2+x}$ is equivalent to $x^2 - x - 2 = 0,\ x > 0$ which by the quadratic formula has a unique solution. Let $y = 2$ be this solution. Then

$$y = \sqrt{2+y} = \sqrt{2 + \sqrt{2+y}} = \cdots = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + y}}}}$$

so y is a solution of the given equation.

y is the unique positive solution, since if z is also a solution, then working backward, z is also a solution of $x^2 - x - 2 = 0,\ x > 0$.

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2. Mixed up socks

Mismatched Joe is in a pitch dark room selecting socks from his drawer. He has only six socks in his drawer, a mixture of black and white. If he chooses two socks, the chances that he draws out a white pair is 2/3. What are the chances that he draws out a black pair?

Solution:

He has a ZERO chance of drawing out a black pair.

Since there is a 2/3 chance of drawing a white pair, then there MUST be 5 white socks and only 1 black sock. The chances of drawing two whites would thus be: 5/6 x 4/5 = 2/3 . With only 1 black sock, there is no chance of drawing a black pair.

3, Minimise this!

Find the minimum value of θ (if it exists) which satisfies

$$y = x\tan\theta + \frac{1}{2} \qquad (0 < \theta < \frac{\pi}{2})$$

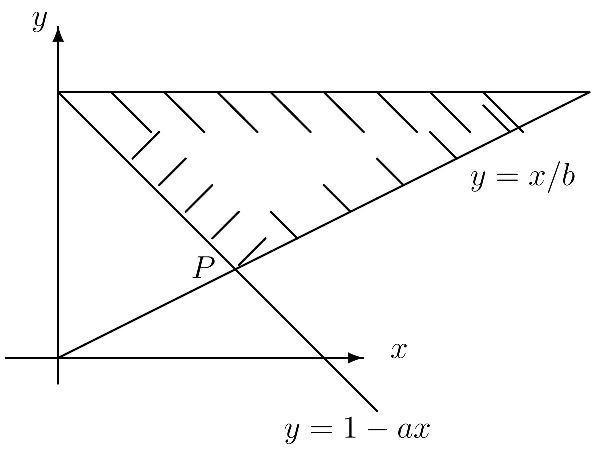
subject to $ax + y \geq 1$

$$0 \leq x \leq by$$

$$0 \leq y \leq 1$$

where a and b are positive.

Solution: The inequality constraints are satisfied by all points (x, y) in the triangular region shown.



The coordinates of P are

$$\left(\frac{b}{1+ab}, \frac{1}{1+ab}\right)$$

The equality constraint is satisfied at all points on a line with positive slope.

If this line meets the shaded region, then all constraints are satisfied.

The y–intercept of the line is $\frac{1}{2}$.

If $\frac{1}{2} > \frac{1}{1+ab}$ then the line passes through the shaded region for any value of θ in $(0, \pi/2)$ so there is no minimum.

If $\frac{1}{2} > \frac{1}{1+ab}$ i.e. $ab < 1$ then the slope of the line is at least
$\dfrac{\frac{1}{1+ab} - \frac{1}{2}}{\frac{b}{1+ab} - 0} = \dfrac{1-ab}{2b}$.

That is,

(a) if $ab > 1$ there is no minimum θ

(b) if $ab < 1$, the minimum value of θ is $\arctan\left(\dfrac{1-ab}{2b}\right)$

(c) if $ab = 1$, $\theta = 0 \notin (0, \pi/2)$.

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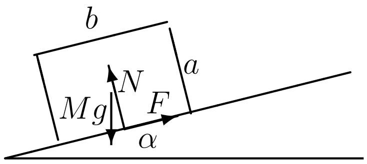
4. Slip or tip?

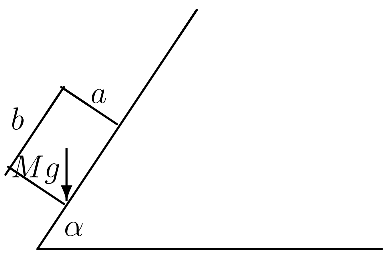
A uniform rectangular block with dimensions $a \times b \times c$ is placed on an inclined plane of small inclination with the c edges perpendicular to the line of greatest slope. The coefficient of friction between the plane and the block is μ.

The inclination to the horizontal is now increased. Will the block slide before it tips, or *vice versa*?

Solution:



Block about to slide



Block about to tip

When at rest, the gravitational force on the block can be regarded as a single force Mg vertically downward through the centre of the block. Likewise the normal reaction of the plane is a single force N perpendicular to the plane passing through the centre of the block. The friction force F acts upwards in the line of the plane.

When at rest, the equilibrium eqations are:

$$\begin{aligned} Mg\sin\alpha - F &= 0 \\ Mg\cos\alpha - N &= 0 \\ F &\le \mu N \end{aligned}$$

where $\tan\alpha \leq \mu$.

If the block has not tipped, it will slip when $\tan\alpha = \mu$.

If the block has not slipped, it will tip when the weight force has positive moment about the lower c edge of the block. The limiting point is reached when $\tan\alpha = b/a$.

Hence the block will slide first if $\mu < b/a$ and tip first if $\mu > b/a$.

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5. Breaking Sticks

Two points are selected independently at random on a straight line segment of length 1. What is the probability that a triangle can be constructed out of these three segments?

Solution:

Let x_1, x_2 and x_3 stand for the lengths of the three segments (one or two of them may be 0.) The three segments serve as the sides of a triangle iff

$$\begin{aligned} x_1 &< x_2 + x_3 \\ x_2 &< x_1 + x_3 \\ x_3 &< x_1 + x_2 \end{aligned} \qquad (1)$$

From the conditions of the problem, $x_1 + x_2 + x_3 = 1$. To each of the inequalities above add its left-hand side to obtain

$$x_1 < 0.5, \qquad x_2 < 0.5, \qquad \text{and } x_3 < 0.5. \qquad (2)$$

The argument is obviously reversible. Therefore also, (2) implies (1). Let us consider the triple (x_1, x_2, x_3) as the barycentric coordinates of a point relative to some fixed triangle ABC. Points that satisfy (2) lie inside the triangle $M_aM_bM_c$, where M_a, M_b, M_c are the midpoints of the sides BC, AC, and AB, respectively. The area of the triangle $M_aM_bM_c$ is one fourth that of ABC. Which shows that $1/4$ is the sought probability.

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6 Maturing Spanish sherry

Spanish sherry is matured as follows. After fermenting in stainless steel vats for two years, it enters the solera system. The solera has barrels in three layers on top of each other. At the end of each year, one third of the wine in the bottom layer is drawn off and bottled, and one third of the wine in each of the top two layers is added to the layer below. The top layer is refilled from the stainless steel vat.

What is the average age of the bottled product?

Solution: Assume instantaneous mixing. After a steady state is reached, let M_i be the average age of the wine in layer i.

Then just before decanting, level 1 has one third of age 3 (2 years in the steel vats and 1 year in the barrel) and two thirds of age $M_1 + 1$. Hence $M_1 = 1 + 2/3(M_1 + 1)$, so $M_1 = 5$.

Similarly, level 2 has one third of age $M_1 + 1$ and two thirds of age $M_2 + 1$. Hence $M_2 = 2 + 2/3(M_2 + 1)$, so $M_2 = 8$.

Then level 3 has one third of age $M_2 + 1$ and two thirds of age $M_3 + 1$. Hence $M_3 = 3 + 2/3(M_3 + 1)$, so $M_3 = 11$. Hence average age in the bottle is 11 years.

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7. Partial fractions

Every expression of the form

$$\frac{1}{(x + a_1)(x + a_2) \cdots (x + a_n)}$$

has a partial fraction decomposition

$$\sum_{i=1}^{n} \frac{A_i}{x + a_i}.$$

Show that if $0 < a_1 < \cdots < a_n$ then the numbers A_i alternate in sign.

Solution: We prove by induction on $n \geq 2$ that the coefficients A_i alternate in sign.

The statement is true for $n = 2$, since $A_1(x-a_2)+A_2(x-a_2) = 1$ and hence $A_1 + A_2 = 0$.

Assume the statement true for $n - 1$ factors $x - a_i$. Then

$$\frac{1}{(x + a_1)(x + a_2) \cdots (x + a_n)} = \left(\sum_{i=1}^{n-1} \frac{B_i}{x + a_i} \right) \frac{1}{x + a_n}$$

where the B_i alternate in sign.

Let $\dfrac{B_i}{(x+a_i)(x+a_n)} = \dfrac{A_i}{x+a_i} + \dfrac{C_i}{x+a_n}$ where each A_i differs in sign from C_i. Then

$$\frac{1}{(x+a_1)(x+a_2)\cdots(x+a_n)} = \sum_{i=1}^{n-1} \frac{A_i}{x+a_i} + \frac{\sum_i C_i}{x+a_n}$$

where the A_i differ in sign and it remains to show that $\sum_i C_i$ differs in sign from A_{n-1}. But this follows from the fact that $a_n > a_{n-1}$.

Alternative Solution Let $\prod_{i=1}^n \dfrac{1}{x+a_i} = \sum_{i=1}^n \dfrac{A_i}{x+a_i}$.

Then $\prod_{i=1}^{n-1} \dfrac{1}{x+a_i} - \prod_{i=2}^n \dfrac{1}{x+a_i} = (a_n - a_1) \prod_{i=1}^n \dfrac{1}{x+a_i}$ (expressing them over the common denominator).

But also $\prod_{i=1}^{n-1} \dfrac{1}{x+a_i} - \prod_{i=2}^n \dfrac{1}{x+a_i} = \displaystyle\sum_{i=1}^{n-1} \frac{B_i}{x+a_i} - \sum_{i=2}^{n-1} \frac{C_i}{x+a_i}$, where, by induction, the B_i and the C_i alternate in sign, both starting with positive.

By the uniqueness of partial fraction decomposition, $B_1 = (a_n - a_1)A_1$, $C_n = (a_n - a_1)A_n$ and for $i = 2, \ldots, n-1$, $B_i - C_i = (a_n - a_1)A_i$. Hence the A_i are alternately positive and negative, as required.

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8. Another triangle inequality

Find the largest k for which

$$|\mathbf{u} - \mathbf{v}| + |\mathbf{v} - \mathbf{w}| + |\mathbf{w} - \mathbf{u}| \geq k(|\mathbf{u}| + |\mathbf{v}| + |\mathbf{w}|)$$

for all be vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^2$ with $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$.

Give a geometric interpretation.

Solution: Since $\mathbf{u}+\mathbf{v}+\mathbf{w} = 0$, $3\mathbf{u} = (\mathbf{u}-\mathbf{v})+(\mathbf{u}-\mathbf{w})$, $3\mathbf{v} = (\mathbf{v}-\mathbf{w})+(\mathbf{v}-\mathbf{u})$, and $3\mathbf{w} = (\mathbf{w}-\mathbf{v})+(\mathbf{w}-\mathbf{u})$. By the triangle inequality,

$$3(|\mathbf{u}| + |\mathbf{v}| + |\mathbf{w}|) \leq 2|\mathbf{u} - \mathbf{v}| + |\mathbf{v} - \mathbf{w}| + |\mathbf{w} - \mathbf{u}|$$

so $k \geq 3/2$ and this value is attained when $\mathbf{u} = \mathbf{v}$ and $\mathbf{w} = -2\mathbf{u}$.

A geometric interpretation is that the sum of the lengths of the medians in a triangle is at least 3/4 of the perimeter.

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9. An Elusive Quadratic

Find a second-degree polynomial with integer coefficients, $p(x) = ax^2 + bx + c$, such that $p(1)$, $p(2)$, $p(3)$, and $p(4)$ are perfect squares (that is, squares of integers), but $p(5)$ is not.

Solution: Trying for a solution that is symmetric about 2.5 and vanishes at 2 and 3 leads to $p(x) = k(x-2)(x-3)$, and we need to find a number k such that $p(1)$ and $p(4)$ are perfect squares, but $p(5)$ is not.

Trial and error quickly leads to $p(x) = 2(x-2)(x-3)$.

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10. A strange continuous function

Let f be a continuous real function such that $f(2004) = 2003$ and for all x, $f(x)f(f(x)) = 1$.

What is $f(2000)$?

Solution: Since $f(f(x)) = \dfrac{1}{f(x)}$, $f(y) = \dfrac{1}{y}$ whenever y is in the range of f.

Since $f(2004) = 2003$, 2003 is in the range of f so $f(2003) = \dfrac{1}{2003}$. Hence $\dfrac{1}{2003}$ is in the range of f as well as 2003. Since f is continuous, by the intermediate value theorem, 2000 is in the range of f.

Hence $f(2000) = \dfrac{1}{2000}$.