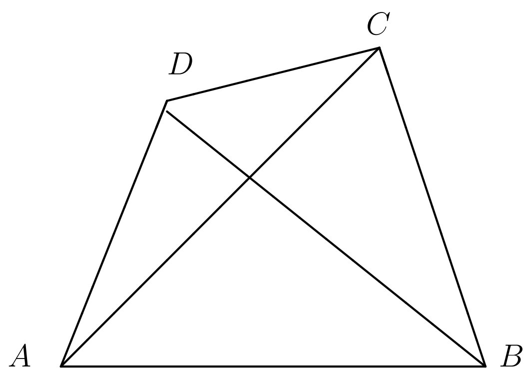
1. Area of a quadrilateral

It is said that tax gatherers in Ancient Egypt estimated the size of a quadrilateral as the product of the averages of the pairs of opposite sides. Show that this method never underestimates the area, and find necessary and sufficient conditions for the estimate to be the correct area.

Solution



If the quadrilateral is not convex, we can construct a convex one of greater area and the same side length by reflecting the ear in a diagonal. Hence if convex quadrilaterals never underestimate area, all quadrilaterals never underestimate area. So we can assume the quadrilateral is convex with area E and denote its vetrices in anti–clockwise order as A, B, C and D as shown.

Draw diagonal AC. Then $E =$ area $ABC +$ area $ACD \leq 1/2(AB.BC+AD.DC)$ with equality if and only if angle $ABC =$ angle $ADC = 90°$. Draw diagonal BD. Then $E =$ area $ABD+$ area $BDC \leq 1/2(BC.CD + DA.AB)$ with equality if and only if angle $BCD =$ angle $DAB = 90°$.

Hence $4E \leq (AB + DC)(BC + DA)$, as required, with equality if and only if $ABCD$ is a rectangle.

2. Student's dream

A common Calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$.

Let $f(x) = e^{x^2}$ and find an open interval (a, b) and a non–zero function g defined on (a, b) such that this false product rule holds true for all $x \in (a, b)$.

Solution

Suppose there exists such a function g, so that

$$f'(x)g(x) + f(x)g'(x) = f'(x)g'(x).$$

Then

$$\frac{g'(x)}{g(x)} = \frac{-f'(x)/f(x)}{1 - f'(x)/f(x)}$$

Since $f(x) = e^{x^2}$, we have

$$\frac{g'(x)}{g(x)} = \frac{-2x}{1 - 2x}$$

so $\ln|g(x)| = x + \frac{1}{2}\ln|1 - 2x| + C$ for an arbitrary constant C.

So we can take any (a, b) with $1/2 < a < b$ and $g(x) = Ae^x\sqrt{2x - 1}$, $A > 0$ or any (a, b) with $a < b < 1/2$ and $g(x) = Ae^x\sqrt{1 - 2x}$, $A > 0$

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3. A sad story

A group of young people consists of three girls and three boys. Each of them is in love with one opposite–sex member of the group. Assume that for each person, each of the three opposite sex persons is equally likely to be the one he or she is in love with. One day, one of the group remarks on the sad fact that no one is loved by the person she or he loves. What is the probability of such a situation?

Solution (Based on the solution of Roger Li, Y2, UWA):

There are altogether $3^6 = 729$ combinations how the young people can fall in love with each other. 156 of these combinations are such that nobody is loved by the person she or he loves. Thus, the probability of such a configuration is

$$\frac{156}{729} = \frac{52}{243} \approx 0.214$$

The 156 combinations in which nobody is loved by the person he or she loves can be found as follows by looking at the $3^3 = 27$ ways the boys can divide their love among the girls:

Case 1 : All three boys love the same girl (this happens in 3 out of the 27 cases). But that girl must love one of the boys and, hence, the sad situation cannot arise.

Case 2 : Two boys love the same girl and the third boy loves one of the remaining two girls (this happens in 18 out of the 27 cases; namely there are three ways of splitting the boys into a 'pair' and 'single', three choices for the 'pair' (or 'single') and two choices for the 'single' (or 'pair')). For the sad situation to occur, the girl that is loved by two boys must love the third boy, the girl who is loved by the third boy must love one of the other two boys and the girl who is not loved by any of the boys can love any of the three boys. Hence, $1 \times 2 \times 3 = 6$ of the 27 ways of how the girls can divide their love among the boys will lead to the sad situation.

This gives $18 \times 6 = 108$ combinations in which the sad situation occurs.

Case 3 : All three boys love a different girl (this happens in 6 out of the 27 cases). Now, the sad situation occurs if each girl loves one of the two boys who does not love her. That is in $2^3 = 8$ out of the 27 ways how the girls can divide their love among the boys, will the sad situation occur.

This gives a further $6 \times 8 = 48$ combinations in which the sad situation occurs.

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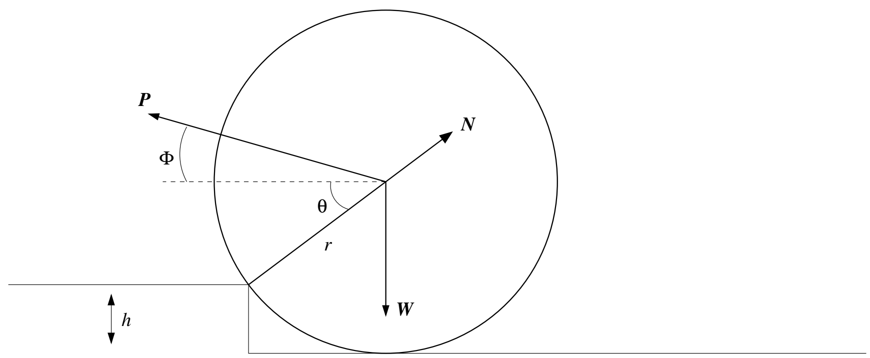
4. Pushing a wheel–chair up the kerb

A wheel of radius r is to be pulled over a kerb of height $h < r$ by means of a force P applied at the axis of the wheel.

(a) Find the least value of P required if P acts horizontally

(b) Find the least possible value of P.

Solution



The external forces on the wheel are its weight W, the required force P acting horizontally in (a) and at an angle ϕ to the horizontal in (b), and the contact forces at the kerb K and the ground G. The contact force at G may be disregarded since it disappears at the instant that the wheel starts to move. The problem is now one of static equilibrium under three forces, so the contact force at K must pass through the centre of the wheel C.

In (a) taking moments about the kerb point,

$$Pr\sin\theta = Wr\cos\theta \Rightarrow P = W\cot\theta = W\frac{\sqrt{2rh-h^2}}{r-h}.$$

Alternatively, resolution of forces gives $P = N\cos\theta$, $W = N\sin\theta \Rightarrow P = W\cot\theta$.

In (b) taking moments about the kerb point,

$$P\cos\phi r\sin\theta = (W - P\sin\phi)r\cos\theta \Rightarrow P = W\frac{\cos\theta}{\sin(\theta+\phi)}$$

which is a minimum when $\theta + \phi = \pi/2 \Rightarrow P = W\cos\theta = W\sqrt{2rh-h^2}/r$.

(The direction of P is then perpendicular to the line joining the kerb point to the centre of the wheel.) Alternatively, this result can be obtained by resolution of forces and elimination of N.

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5. Shooting Stars

Henry and Gretchen plan on sitting outside to look for shooting stars. They know from experience that if they watch for an hour, they will have a 90% chance of seeing at least one shooting star.

Their experience also tells them that the number of shooting stars seen in any given time interval is independent of the number of shooting stars seen in any other disjoint time interval. Also, the average number of shooting stars seen in a given time interval is a constant that depends only on the length of the time interval.

It is a chilly night, though, so Gretchen says, "Let's only stay out for 10 minutes".

Henry says, "I was really hoping to see a shooting star tonight. If we are only out for 10 minutes, we will only have a 15% chance".

Gretchen replies, "Not true. We have a better chance than that".

Is Gretchen right? If so, what is the probability that they see a shooting star?

Solution

Gretchen is right. The probability that they will see at least one shooting star is about 32%.

We know that the probability that they do not see a shooting star over the course of an hour is 10%. This is the product of not seeing a shooting star for 6 consecutive 10-minute periods. So if q is the probability of not seeing a shooting star over a 10-minute period, we can say: $0.1 = q^6$ or $q = 0.6813$.

We know that the probability that they do see at least one shooting star is just 1 minus the probability that they do not see any, or 1 - 0.6813, which equals about 32%.

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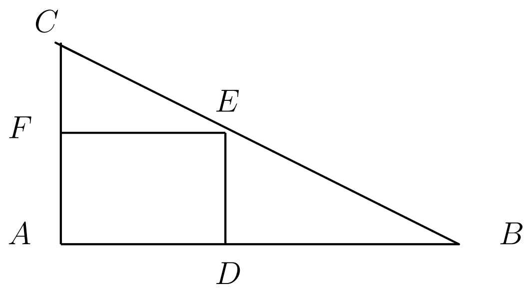
6. Largest rectangle in a triangle

In a Calculus exam, students were asked to find the largest rectangle that would fit inside a right–angled triangle. Sam thought that the largest rectangle would be one whose longest side lay along the longest side of the right angle. Pam thought that the largest rectangle would be one whose longest side lay along the hypotenuse.

Who was right, and by the way could there be an even larger rectangle that would fit inside the triangle?

Solution (based on the solution of Evgeni Sergeev, Y2, UWA)

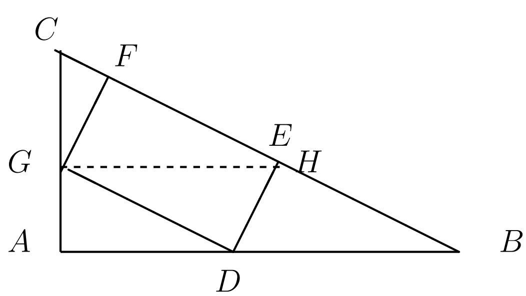
Clearly any largest rectangle must have points of contact with each of the sides of the triangle, otherwise it could be enlarged.



In Sam's solution, the triangles ABC, DBE and FEC are similar, say $DB = kAB$, $DE = AF = kAC$.

The triangles FEC and ABC are similar, and moreover $FC + DE = AC$, so $FC = (1-k)AC$ and $FE = AD = (1-k)AB$.

Let $\Delta = AB\,.\,AC$ be twice the area of the triangle ABC. Then area of the rectangle $= AF\,.\,AD = k(1-k)\Delta$. This clearly reaches its maximum when $k = 1/2$, so the largest rectangle according to Sam has area 1/4 the area of ABC.



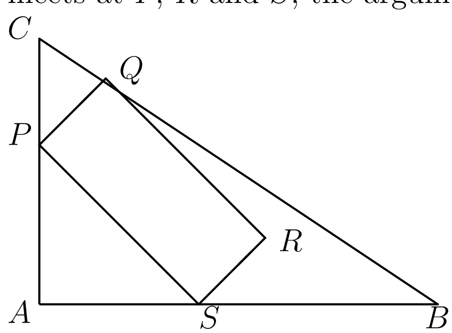
In Pam's solution, the line GH is supposed to be parallel to AB so that triangle GFH is congruent to triangle DEB and hence the area of the required rectangle is the area of the parallelogram $DGHB$, that is $DB\,.\,AG$.

The triangles AGD and ACB are similar, say $AG = \ell AC$ and $AD = \ell AB$. So $DB = AB - AD = (1-\ell)AB$.

Let $\Delta = AB\,.\,AC$ be twice the area of the triangle ABC. Then area of the parallelogram $= DB\,.\,AG = \ell(1-\ell)\Delta$. This clearly reaches its maximum when $\ell = 1/2$, so the largest rectangle according to Pam has area $1/4$ the area of ABC.

Thus Pam's maximum rectangle has the same area as Sam's (but in fact has a different shape).

In general, a maximal rectangle must have at least one corner on each side of the triangle, otherwise it could be enlarged. Suppose a rectangle $PQRS$ meets the sides of the triangle at P, Q and S as shown. Such a rectangle cannot be maximal, because a small rotation about Q would leave it contacting the triangle only at Q and hence it can be enlarged. If instead it meets at P, R and S, the argument is similar.



So both Pam and Sam are correct, and furthermore, this is the largest possible rectangle in a triangle.

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7. A matrix operation

Let A and B be $n \times n$ matrices over the reals. Define $A \circ B$ to be the matrix $A + B - AB$. Find necessary and sufficient conditions on A for the equation $A \circ B = B \circ A = \mathbf{0}_{n \times n}$ to have a solution B.

Solution The condition is that $I_n - A$ is non–singular. First note that $A \circ B = \mathbf{0}$ if and only if $(I_n - A)(I_n - B) = I_n$. So if $A \circ B = \mathbf{0}$ then $I_n - A$ is non–singular.

Conversely, if $I_n - A$ is non–singular, let C be its inverse and let $B = I_n - C$. Then $C = I_n - B$ so $(I_n - A)(I_n - B) = I_n$

8. Flexible quadrilateral

A machine linkage for modifying rotary motion has the form of a quadrilateral $Q = ABCD$ made of four rigid rods with flexible joints at the vertices. The linkage can move freely subject to the following constraints:

(a) Q remains in a fixed plane.

(b) Rod AB is fixed in the plane while rods BC, CD and DA can rotate in either sense about their end-points.

(c) All rod lengths remain constant (not necessarily the same constant).

Show that if at any time the two diagonals AC and BD are perpendicular, they are always perpendicular.

Solution Let $\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3$ and $\mathbf{a}_4$ be vectors representing the rods in anticlockwise order so that their lengths are fixed but their directions may vary, subject to $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0}$. It follows that $|\mathbf{a}_1 + \mathbf{a}_3| = |\mathbf{a}_2 + \mathbf{a}_4|$ and hence that

$$|\mathbf{a}_2|^2 + |\mathbf{a}_4|^2 - |\mathbf{a}_1|^2 - |\mathbf{a}_3|^2 = 2(\mathbf{a}_1 \cdot \mathbf{a}_3 - \mathbf{a}_2 \cdot \mathbf{a}_4).$$

Let the diagonals be $\mathbf{d}_1 = \mathbf{a}_4 + \mathbf{a}_1$ and $\mathbf{d}_2 = \mathbf{a}_3 + \mathbf{a}_4$. Then $\mathbf{d}_1 \cdot \mathbf{d}_2 = \mathbf{a}_4 \cdot \mathbf{a}_3 + \mathbf{a}_4 \cdot \mathbf{a}_4 + \mathbf{a}_1 \cdot \mathbf{a}_3 + \mathbf{a}_1 \cdot \mathbf{a}_4 = \mathbf{a}_1 \cdot \mathbf{a}_3 + \mathbf{a}_4(\mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_1) = \mathbf{a}_1 \cdot \mathbf{a}_3 + \mathbf{a}_4 \cdot (-\mathbf{a}_2) = \mathbf{a}_1 \cdot \mathbf{a}_3 - \mathbf{a}_2 \cdot \mathbf{a}_4 = \frac{1}{2}(|\mathbf{a}_2| + |\mathbf{a}_4| - |\mathbf{a}_1| - |\mathbf{a}_3|)$, which is constant.

Hence if for some configuration $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$, then $|\mathbf{a}_2| + |\mathbf{a}_4| = |\mathbf{a}_1| + |\mathbf{a}_3|$ so $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$ in all configurations.

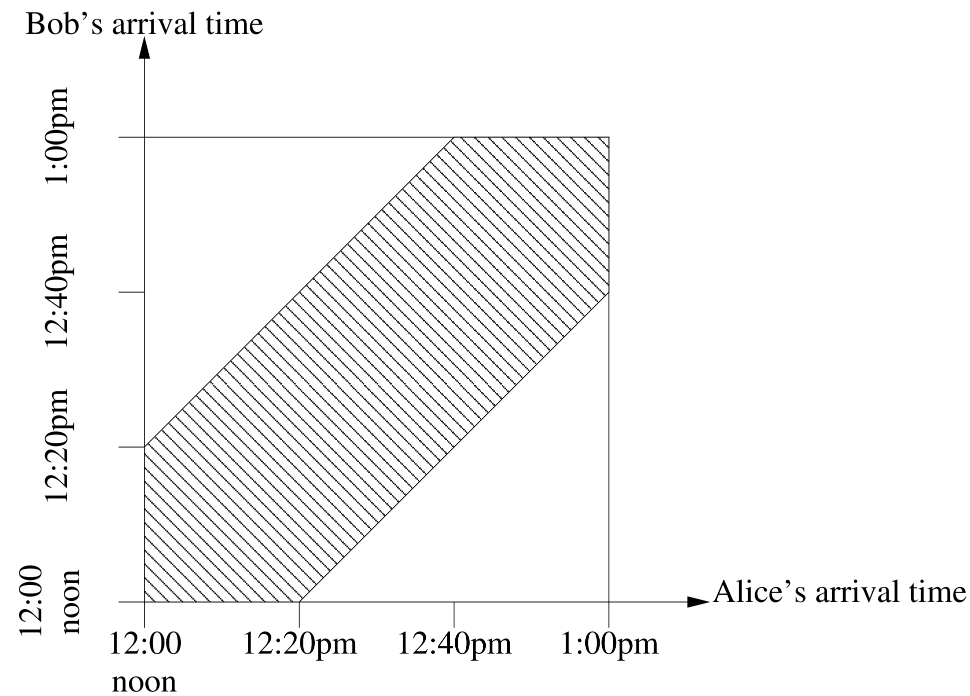
9. Meeting in the park

Alice and Bob agree to meet in the park. They agree to each show up at some random time between 12:00 noon and 1:00 PM, and wait 20 minutes for the other person or until one o'clock, whichever comes first.

Assuming that they stick to their word, what is the probability that they will meet?

Solution

Their probability of meeting is $\frac{5}{9}$.



The solution can be found via the above graphic which illustrates the arrival times of Alice and Bob. They will meet if their joint arrival times fall into the shaded area. Since any pair of arrival times is equally likely, the probability of Alice and Bob meeting is equal to the fraction of the area that the shaded polygon occupies. Simple geometry shows that the joint area of the two triangles (unshaded regions) is $\frac{4}{9}$ of the total area, whence the area of the shaded region is $\frac{5}{9}$ of the total area.

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10. The island

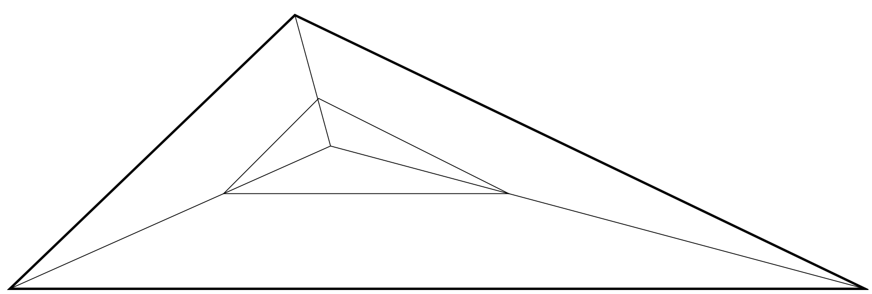
An island has the form of a polygonal convex region on a flat earth. In times of emergency civilians are not allowed within a distance d of the coast, where the value of d depends on the emergency. The capital (modelled as a point) is to be sited so as to lie in all possible regions accessible to civilians.

(a) Determine the location of the capital if the island is a triangle.

(b) If the island is a quadrilateral, but not a trapezium, determine the location of the capital.

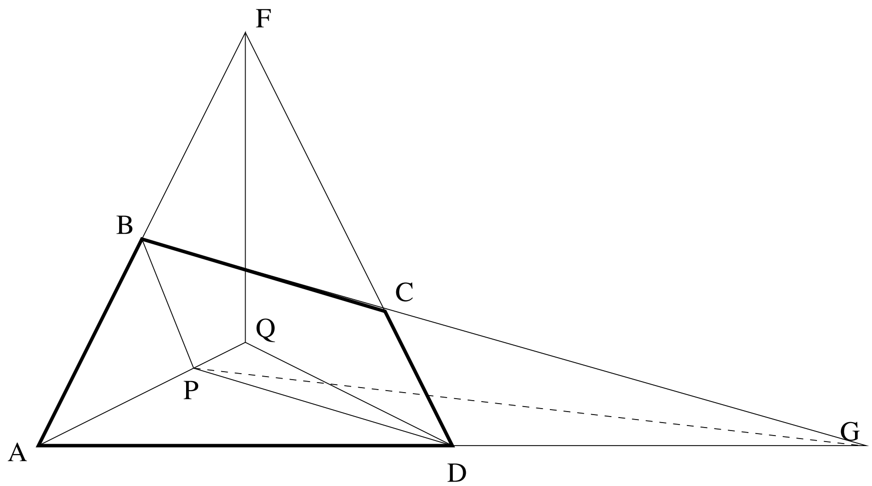
(c) Where should the capital be if the island is a trapezium?

Solution (a) A point on the internal bisector of an angle of the triangle is equidistant from the sides. It follows that the region accessible to civilians for a given (small) value of d is a triangle similar to the given triangle whose vertices lie on the internal bisector of the angles. To cope with all values of d the capital must be sited as far as possible from any coastline. Hence the capital is at the point of intersection of the internal bisectors, i.e. the in-centre of the triangle. (If the distance from each side is denoted $d_{\max}$, there are points further from two sides than $d_{\max}$, but then the distance from the third side is less than $d_{\max}$.)



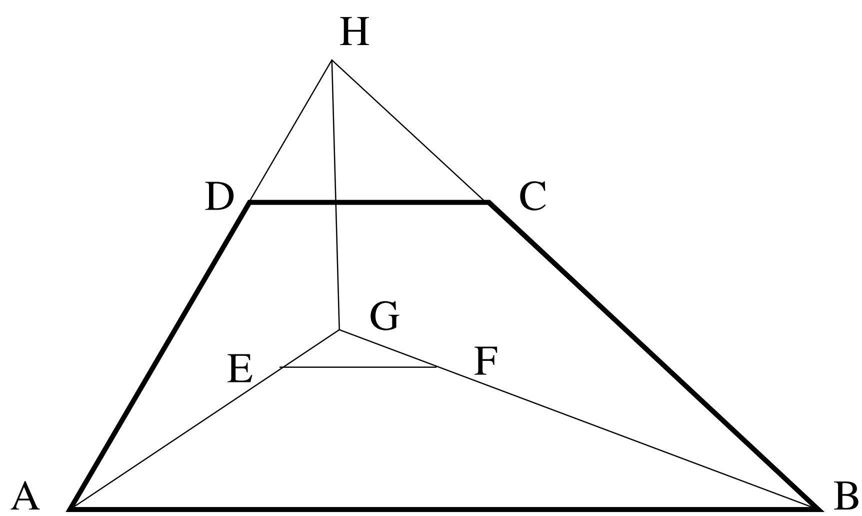
(b) Since the quadrilateral is convex, the internal angles must be less than π, and hence there exists an unambiguous internal bisector at each vertex.. Suppose the labelling of the vertices is such that when produced AB and DC meet at F, and AD and BC meet at G. In the diagram P, the in-centre of $\Delta\, ABG$, is equidistant from AB, AD and BC, while Q, the incentre of ΔADF, is equidistant from $AB,\ AD$, and CD.

Thus P, the in-centre closer to A, the common vertex of the two triangles, is the same distance from three sides (and is as large as it can be) while the distance from the fourth side is greater, so that P satisfies the conditions for location of the capital.



(c) Let AB and CD be parallel and suppose the labelling of the vertices is such that when AD and BC are produced they meet at H. Let the distance between AB and CD be Δ and let EF be parallel to AB and mid-way between AB and CD. Let the bisectors of the angles of the triangle ABH meet at G.

Case 1. EF meets two of the bisectors in E and F respectively. Then the distance from E to $AB,\ AD$ and CD is $\Delta/2$, as is the distance from F to $AB,\ BC$ and CD. The distance from any point in EF to any of the sides is equal to or greater than $\Delta/2$ and $\Delta/2$ is the maximum distance any point can be from all four sides. Hence the capital could be anywhere in EF.



Case 2. EF meets only one bisector. The distance from G to AB, BC and AD is $\delta \leq \Delta/2$.

The distance of any point other than G to at least one of the sides is less than δ. Hence the distance from G to any of the sides is equal to or greater than δ and the capital should be at G.