The Blakers Mathematics Contest 2008 Solutions

1. **An even sum** Let $n \geq 2$ and let $a_1, a_2, \ldots, a_n$ be integers.
   Prove that

$$|a_1 - a_2| + |a_2 - a_3| + \cdots + |a_n - a_1|$$

is even.

Solution (Y. Choong, 1st year, UWA): First note that the sum $a_1 - a_2 + a_2 - a_3 + \cdots + a_n - a_1 = 0$ is even.

Now note that replacing each $a_i - a_{i+1}$ (where $a_{i+1} = a_1$ if $i = n$) by $|a_i - a_{i+1}|$ does not change the parity of the sum, for if $a_i \geq a_{i+1}$ the sum is unchanged, while if $a_i < a_{i+1}$, the sum is increased by $2(a_{i+1} - a_i)$.

Hence the parity of $|a_1 - a_2| + |a_2 - a_3| + \cdots + |a_n - a_1|$ is the parity of 0, namely even.

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2. **Who Stole the Book?**

Six students visited the library on the day a rare book was stolen. Each student entered once, stayed for some time, and left. For any two of them that were in the library at the same time, at least one of them saw the other. The dean questioned the students and learned the following:

Student	Reported seeing
Alice	Bob and Eve
Bob	Alice and Frank
Charlie	Doris and Frank
Doris	Alice and Frank
Eve	Bob and Charlie
Frank	Charlie and Eve

The dean believes that each student reported all the others that he or she saw, with the exception of the thief who, in an attempt to frame another student, reported that other student as being seen when that other student was not in fact in the library. Assume the dean's belief is correct. Who stole the book?

Solution: (Adapted from Y. Choong, 1st year, UWA) Consider the directed graph whose vertices are he students with arrows from x to y if x reported seeing y. Then the conditions imply that any undirected cycle of length 4 or more of truth tellers must have a diagonal. For example, if A saw E saw C saw D saw A, then either A and C, or E and D were in the library together.

Hence any undirected 4-cycle without diagonals must contain the liar. The only 4-cycles without diagonals are $ADFB$, $ADFE$ and $AECD$. Thus the thief is either Alice or Doris. But it cannot be Alice, since if both her statements are ignored, there would still be a 4-cycle $ABFD$ without diagonals. Hence Doris is the thief and removing the arrow AD would eliminate all 4-cycles without diagonals.

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3. Idempotent matrices

An $n \times n$ real matrix P is called an idempotent if $P^2 = P$. Show that if A, B and C are idempotents with $A+B+C = 0_n$ then they are all 0_n.

[Hint: First show that an idempotent matrix acts as the identity matrix on vectors in its range.]

Solution: (Adapted from Wilson Ong, 3rd year, UWA) If a vector $\mathbf{y} \in \text{range}(P)$ of an idempotent matrix P, say $\mathbf{y} = P(\mathbf{x})$ then $P(\mathbf{y}) = P^2(\mathbf{x}) = P(\mathbf{x}) = \mathbf{y}$.

Let $\mathbf{u}$ be in the range of A so that $A\mathbf{u} = \mathbf{u}$. Let $\mathbf{v} = B\mathbf{u}$ so $B\mathbf{v} = \mathbf{v}$.

Then $C\mathbf{u} = -A\mathbf{u} - B\mathbf{u} = -\mathbf{u} - \mathbf{v}$, so $\mathbf{u} + \mathbf{v} = C(-\mathbf{u}) = C^2(-\mathbf{u}) = C(\mathbf{u} + \mathbf{v})$ and hence $C\mathbf{v} = 2(\mathbf{u} + \mathbf{v})$.

Hence $A\mathbf{v} = -B\mathbf{v} - C\mathbf{v} = -2\mathbf{u} - 3\mathbf{v}$. Since A is idempotent, it follows that $-2\mathbf{u} - 3\mathbf{v} = -2A\mathbf{u} - 3A\mathbf{v} = 4\mathbf{u} + 9\mathbf{v}$. Consequently, $\mathbf{u} = -2\mathbf{v}$ and hence $B\mathbf{u} = -\mathbf{u}/2$. Since B is an idempotent, this implies that $\mathbf{u} = 0$.

Since $\mathbf{u}$ was defined to be any element in the range of A, A must be 0, and similarly for B and C.

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4. A differential inequality

Let $x(t)$ be a twice differentiable function on $0 < t < \pi/2$ such that $x(0) = 0$ and $\dot{x}(0) = 1$. Suppose that $\ddot{x} + x \geq 0$.

Prove that $x \geq \sin(t)$.

Is the converse true?

[Hint: Note that $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right)$]

Solution:

$$\begin{aligned}\ddot{x} + x &= \frac{d}{dx}(\dot{x}^2 + x^2)/2 \geq 0 \\ &\Rightarrow (\dot{x}^2 + x^2)/2 \geq (1+0)/2 \\ &\Rightarrow \dot{x}^2 \geq 1 - x^2 \\ &\Rightarrow \dot{x} \geq \sqrt{1-x^2} \\ &\Rightarrow \int_0^t \frac{\dot{x}}{\sqrt{1-x^2}}\, dt \geq \int_0^t 1\, du = t\end{aligned}$$

Hence $\sin^{-1} x \geq t$, so $x \geq \sin t$.

Alternatively, let $\ddot{x} + x = f(t)$, where $f(t) \geq 0$

Solution by variation of parameters or Laplace transforms yields

$$x = \sin\, t + \int_0^t f(u) \sin(t-u)\, du \geq \sin\, t$$

since the integrand and hence the integral is positive.

The converse :

(A) $$x \geq \sin(t) \text{ with } x(0) = 0 \text{ and } \dot{x}(0) = 1,\ 0 < t < \pi/2$$

implies

(B) $$\ddot{x} + x \geq 0$$

is false.

Consider $x(t) = \sin t + (1 - \cos 2t)$.This function satisfies (A). But $\ddot{x} + x = 3\cos 2t + 1$, which is negative for t close to $\pi/2$, say $t = 1$, so (B) is not true.

Alternative Solution to inequality(Callum Shakespeare, Year 1, UWA)

Let $g(t) = \dfrac{x(t)}{\sin t}$, so that $g'(t) = \dfrac{\sin t\, \dot{x}(t) - x(t)\cos t}{\sin^2 t} = \dfrac{h(t)}{\sin^2 t}$

Then $h'(t) = \sin t(\ddot{x}(t) + x(t)) \geq 0$ on $(0, \pi/2)$,

Since $h(0) = 0$ and $h(t)$ is increasing on $(0, \pi/2)$, $g'(t) = \dfrac{h(t)}{\sin^2 t}) \geq 0$ on $(0, \pi/2)$.

Now $\lim_{t\to 0^+} g(t) = \lim_{t\to 0^+} \dfrac{x'(t)}{\cos t} = 1$ by l'Hospital's Rule. Hence $g(t) \geq 1$, so $x(t) \geq \sin t$ on $(0, \pi/2)$

5. Complex numbers

Prove that $|1+z| \leq |1+z|^2 + |z|$ for all complex numbers z

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Solution (Jonathan Blanchard, Year 3, Curtin)
$(|z+1|-1)^2 = |z+1|^2 - 2|z+1| + 1 \geq 0$ so

$|z+1|^2 + |z| \geq 2|z+1| + |z| - 1$

By the triangle inequality, $|z+1| + |z| \geq 1$ so that
$2|z+1| + |z| - 1 \geq |z+1|$

Hence $|z+1|^2 + |z| \geq |z+1|$.

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6. Black and white graphs

Prove that for every finite graph, you can colour the vertices black or white so that at least half the neighbours of of every white vertex are black and at least half the neighbours of every black vertex are white.

Solution Among all the finitely many possible black and white colourings, choose one that maximises the total number of adjacent pairs of oppositely coloured vertices.

This colouring has the required property. If not, either there is a white vertex v with fewer than half its neighbours black or there is a black vertex v with fewer than half its neighbours white. Changing the colour of v would increase the number of adjacent pairs of oppositely coloured vertices, contradicting maximality.

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7. Permutation maxima

We say a permutation π of $1, 2, \ldots, n$ has a *local maximum* at k if
(a) $\pi(k) > \pi(k+1)$ if $k = 1$
(b) $\pi(k) > \pi(k-1)$ and $\pi(k) > \pi(k+1)$ if $1 < k < n$
(c) $\pi(k) > \pi(k-1)$ if $k = n$.

For example, if $n = 5$ then the permutation $(2, 1, 4, 5, 3)$ has local maxima at $k = 1$ and $k = 4$.

If all permutations are equally likely, what is the expected number of local maxima of π?

Solution (Adapted from Aaran Mohann, Year 1, Curtin) Consider the probability that a random permutation has a local maximum at k. If $k = 1$ or n, this is just $1/2$ since it is equally likely that $\pi(1) < \pi(2)$ or $\pi(1) > \pi(2)$, and similarly for $\pi(n-1)$ and $\pi(n)$.

If $1 < k < n$ then the probability that $\pi(k)$ is the maximum of $\pi(k-1)$, $\pi(k)$ and $\pi(k = 1)$ is $1/3$. The number of such consecutive points is $n-2$. Hence the expected number of local maxima is $1/2 + (n-2)/3 + 1/2 = (n+1)/3$.

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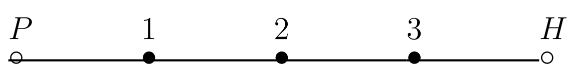
8. A drunkard's walk

A tipsy gentleman leaves the pub to walk home, a distance of four blocks in a straight line.



At each corner, he tosses a fair coin to decide whether to go forwards or backwards.

What is the expected length of his walk?

Solution

Let $e(i)$ denote the expected length of the drunkard's walk from i to H, $i = P$, 1, 2, 3, H, so $e(H) = 0$ and we must find $e(P)$..

1. From P he can only go to 1, so $e(P) = e(1) + 1$ and hence $e(1) = e(P) - 1$

2. From 1 he can go to P with probability 1/2 and to 2 with probability 1/2 so $e(1) = e(P) - 1 = 1/2(e(P) + e(2)) + 1$ and hence $e(2) = e(P) - 4$

3. From 2 he can go to 1 with probability 1/2 and to 3 with probability 1/2 so $e(2) = e(P) - 4 = 1/2(e(1) + e(3)) + 1$ and hence $e(3) = e(P) - 9$ (suggesting $e(i) = e(P) - i^2$)

4. From 3 he can go to 2 with probability 1/2 and H with probability 1/2 so $e(3) = (1/2)e(2) + 1 = (1/2)e(P) - 1$.

5. Hence $e(P) - 9 = (1/2)e(P) - 1$ so $e(P) = 16$

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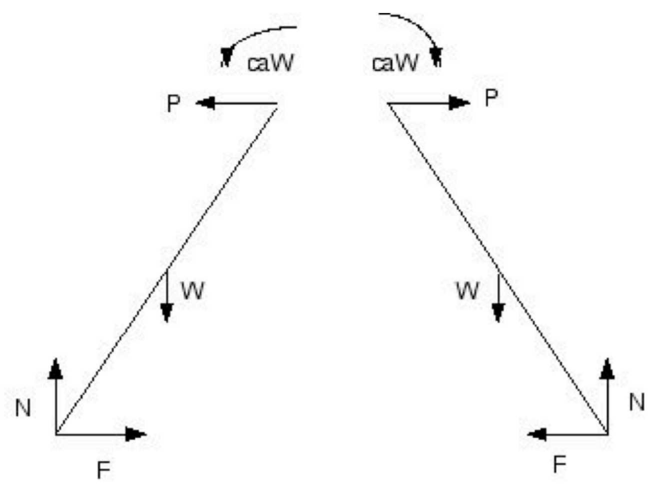
9. The toppling wheelie bin

In an attempt to understand how a 'wheelie–bin' when emptied sometimes is in an inclined position propped up by the lid, consider the following greatly simplified problem. Two rectangular plates of size $2a \times b$ and weight W are hinged along the b–edge. When placed in an inverted V on a rough horizontal plane the included angle is 2θ. The action–reaction between the plates consists of a force and a friction couple caW where c is a constant. How rough must the horizontal plane be so that the plates do not move?

(Roughness is measured by the coefficient of friction μ which relates the friction force F to the normal reaction N).

Solution The diagram below shows the forces on each of the plates. By symmetry, the force exerted on each plate by the other must be horizontal. On each plate the friction couple acts in the direction to oppose any rotation.



Let θ be the angle at the top between the plates and the vertical.

On both plates, horizontally $P = F$ and vertically $W = N$.

Taking moments about the hinge, $N2a\sin\theta = caW + Wa\sin\theta + F2a\cos\theta$.

Then $F2a\cos\theta = Wa\sin\theta - caW$

Hence $F/N = (\tan\theta - c\sec\theta)/2$

For the plates to be stationary, $F \leq \mu N$,
i.e., $\mu \geq (\tan\theta - c\sec\theta)/2$.

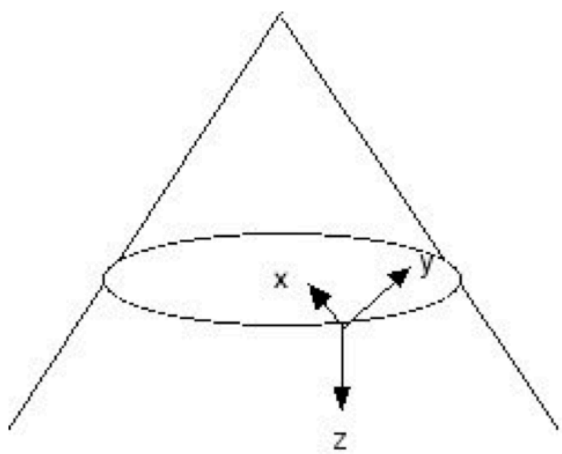
10. Loop on a cone

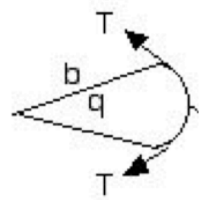
A flexible extensible loop of cable of mass M and length $2\pi a$ when not extended, is placed on a smooth cone, semi–vertical angle ϕ, vertex up, forming a horizontal circle of radius b. The tension in the cable is $2\pi k(b-a)$ [Hooke's Law], where k is the elastic constant, a property of the material.

Determine b.

Solution Consider a small element of the cable subtending an angle $\delta\theta$ at the centre of the circle of radius b.

In the diagram below, q represents the small angle $\delta\theta$, x the force $T\delta\theta$, y the force $N\delta\theta$ and z the force $Mg\delta\theta/2\pi$.





Resolving the tension forces on the element in directions towards the centre and tangential to the arc, the force towards the centre is $2T\sin(\delta\theta/2) \approx T\delta\theta$.

The forces on the element are $Mg\delta\theta/2\pi$ vertically downward, $T\delta\theta$ horizontal toward the centre, and $N\delta\theta$ normal to the surface of the cone.

Resolving in horizontal and vertical directions,

$$2T\sin\frac{\delta\theta}{2} = T\delta\theta = N\delta\theta\cos\phi \text{ and } N\delta\theta\sin\phi = Mg\delta\theta/2\pi.$$

Hence $T = (Mg/2\pi)\cot\phi = 2\pi k(b-a)$ and therefore $b = a + (Mg/4\pi^2 k)\cot\phi$.

Alternative energy solution: (Adapted from Callum Shakespeare, 1st year, UWA)

In the diagram above, consider a general position of the cable where the radius is x.
The distance of the cable below the vertex is $x \cot \phi$.
The gravitational potential energy is $C - Mgx \cot \phi$ where C is an arbitrary constant.
The elastic potential energy is $k(\text{extension})^2/2 = 4k\pi^2(x-a)^2/2$.
The total potential energy $V = C - Mg \cot \phi + 2k\pi^2(x-a)^2$.
At equilibrium, $\frac{dV}{dx} = 0$.
i.e., $\quad -Mg \cot \phi + 4k\pi^2(x-a) = 0$.
i.e., $\quad x = a + Mg \cot \phi / (4\pi^2 k)$
At equilibrium the radius is b, i.e., $\quad b = a + Mg \cot \phi / (4\pi^2 k)$