

The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS
BLAKERS MATHEMATICS COMPETITION

2013 Problems with a solution

1. Planely Euclidean

In the Euclidean plane, the distances from a point p to three vertices of a square are 1, 2 and 3, respectively.

What are the possible side lengths of the square?

Solution. [Best solution by Sahil Khubchandani, 3rd year, UWA]

Synopsis: set up equations, where, without loss of generality, point $P(-s, 0)$ is at distances s, r, t from vertices of square $S(0, 0), R(-h, v), T(-v, h)$, respectively, obtaining

$$t = \sqrt{(h + s)^2 + v^2}$$
$$r = \sqrt{(s - v)^2 + h^2}$$

or equivalently

$$t^2 = h^2 + v^2 + s^2 + 2sh$$
$$r^2 = h^2 + v^2 + s^2 - 2sv$$

which leads to all possible values of the square's sidelength $\ell = \sqrt{h^2 + v^2}$. We now find solutions where (s, r, t) are permutations of $(1, 2, 3)$, noting that some permutations lead to the same solution for ℓ . Essentially, we have three cases:

Case 1: $s = 1, r = 2, t = 3$. Here we have,

$$9 = h^2 + v^2 + 1 + 2h$$
$$4 = h^2 + v^2 + 1 - 2v$$

2. Digitally diverse integers

What positive integers have an integer multiple whose writing in the decimal system includes at least one of each digit 0, 1, 2, 3, 4, 5, 6, 7, 8, 9?

3. Parabolic chords

Take a parabola P and trace all chords between points p, q on P such that the area bounded by the chord and P is some fixed constant k .

Determine the locus C of midpoints of these chords.

4. Empowered primes

Observing that $2 = 3^1 - 2^0$, $3 = 2^2 - 3^0$, $5 = 3^2 - 2^2$, $7 = 3^2 - 2^1$, $11 = 3^3 - 2^4$, $13 = 2^4 - 3^1$, $17 = 3^4 - 2^6$, $19 = 3^3 - 2^3$, it is tempting to conjecture that each prime number is the difference (in absolute value) of integer powers of 2 and 3.

Is this conjecture true or false?

5. Tetra-balls

Four balls of radius 1 are in the interior of a regular tetrahedron. Each ball is tangent to the other three, and each face of the tetrahedron is tangent to three balls.

What is the length of an edge of the tetrahedron?

6. Pointedly coloured

Can we colour each point of the Euclidean plane red, blue or green, in such way that no rectangle has four vertices of the same colour?

7. Determinedly even

The coefficients of a 10×10 matrix are randomly chosen from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

If these choices are independent, what is the probability (with 4 decimal place accuracy) that the determinant of the matrix is even?

8. Really polynomial

For which values of n does there exist a polynomial function $p : \mathbb{C} \rightarrow \mathbb{C}$ of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

with complex coefficients, such that $p(x) \in \mathbb{R}$ if and only if $x \in \mathbb{R}$.

9. Preserving distance

Denote by $d(x, y)$ the usual Euclidean distance between two points $x, y \in \mathbb{R}^n$. If α is a permutation of points of \mathbb{R}^n preserving rational distances, in the sense that, if $d(x, y) \in \mathbb{Q}$ then $d(\alpha(x), \alpha(y)) = d(x, y)$, can we deduce that α preserves all distances (that is, $d(\alpha(x), \alpha(y)) = d(x, y)$ for all $x, y \in \mathbb{R}^n$)

(a) if $n = 1$, that is, on the number line?

(b) if $n = 2$, that is, in the Euclidean plane?

10. Rationally irrational

Is there a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x)$ is irrational for all rational x , and rational for all irrational x ?
