UWA Mathematics Union

Blakers Mathematics Competition 2023

Open to first to third year students of any Western Australian university,
with prizes sponsored by the UWA Mathematics Union.

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers).

2023 Problems with Solutions

1. Quad warm-up exercise

Find the area of the quadilateral of largest area with sides 4, 16, 17, 23.

Solution. First we may suppose that the sides of lengths 4 and 23 are adjoining in the quadrilateral of largest area, since if they are not, we can cut the quadilateral along a diagonal and turn over one of the triangles obtained.
Now, since a triangle with sides a, b and included angle θ has area

$$\frac{1}{2}ab\sin\theta \leqslant \frac{1}{2}ab,$$

with the maximum value achieved when $\theta = 90°$, the quadrilateral's area is bounded above by,

$$\tfrac{1}{2} \cdot 4 \cdot 23 + \tfrac{1}{2} \cdot 16 \cdot 17 = 182,$$

and this is achievable since

$$4^2 + 23^2 = 16^2 + 17^2$$

shows that the right triangle with legs 4 and 23, and the right triangle with legs 16 and 17 can be joined along their hypotenuses to form a quadrilateral.
Thus, the quadilateral of largest area with sides 4, 16, 17, 23, has area: 182.

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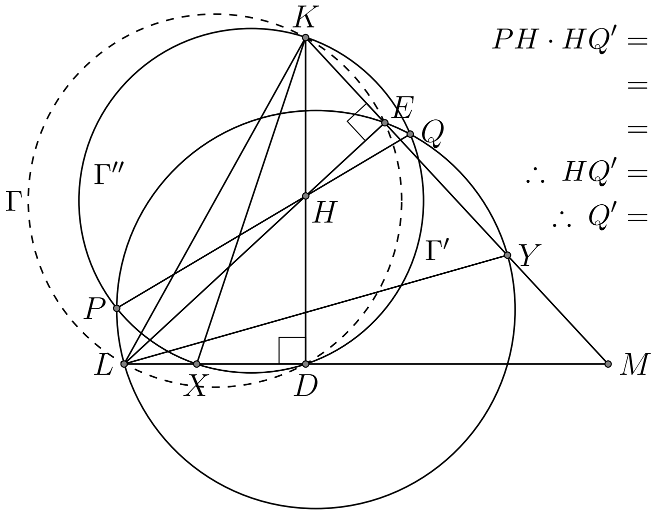
2. Geometrically aligned

Let X, Y be points on sides LM, MK of acute triangle KLM, respectively, and let H be the orthocentre of $\triangle KLM$.
Prove that the points of intersection of the circles with diameters KX and LY, and H are collinear.

Solution. Let Γ' and Γ'' be the circles with diameters KX and LY, respectively, and let P, Q be the points at which Γ' and Γ'' intersect.
Let D, E be the feet of the altitudes of $\triangle KLM$, dropped from K, L respectively. Then

$$\begin{aligned} \angle KDL &= 90° \\ &= \angle KEL, \text{ i.e. angles at } D, E \text{ on } KL \text{ are equal} \\ \therefore\ KLDE &\text{ is cyclic.} \end{aligned}$$

Let Γ be the circle through points K, L, D, E.
Let Q', Q'' be the points (other than P) where PH again meets circles Γ', Γ'', respectively.



$$\begin{aligned} PH \cdot HQ' &= KH \cdot HD, \text{ power of point } H \text{ in } \Gamma' \\ &= LH \cdot HE, \text{ power of point } H \text{ in } \Gamma \\ &= PH \cdot HQ'', \text{ power of point } H \text{ in } \Gamma'' \\ \therefore\ HQ' &= HQ'' \\ \therefore\ Q' &= Q'', \text{ since } Q', Q'' \text{ are same distance from } H, \text{ along ray } PH. \end{aligned}$$

Since Q' on Γ' and Q'' on Γ'' are the same point, they coincide with the point Q.
Therefore, P, H, Q are collinear.

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3. Cyclic divisibility

Let $x, y, z \in \mathbb{N}$ such that x divides y^4, y divides z^4, and z divides x^4.
Prove xyz divides $(x+y+z)^{23}$.

Solution. It's convenient to use the notation "$|$" for *"divides"*.
Expanding $(x+y+z)^{23}$ we get terms of the form $x^k y^\ell z^m$.
We are done if we can show $xyz \mid x^k y^\ell z^m$, for the various non-negative integer possibilities of k, ℓ, m such that $k + \ell + m = 23$.
Due to the cyclic nature of the given divisibilities of x, y, z, we need only consider the following cases.

Case 1: $k, \ell, m \geqslant 1$. Then $xyz \mid x^k y^\ell z^m$, trivially.

Case 2: One of k, ℓ, m is 0, and the others are non-zero.
Without loss of generality, $m = 0$.
First suppose $k \geqslant 5$. Then $x^k y^\ell = x^{k-4} y^\ell x^4$ with

$$x \mid x^{k-4},\ y \mid y^\ell,\ z \mid x^4,$$

so that $xyz \mid x^k y^\ell$, in this case.
Now consider the remaining subcase: $1 \leqslant k \leqslant 4$. Then $\ell \geqslant 19 > 16$, so that $x^k y^\ell = x^k y^{\ell-16} y^{16}$ with

$$x \mid x^k,\ y \mid y^{\ell-16},\ z \mid x^4 \mid (y^4)^4 = y^{16},$$

so that $xyz \mid x^k y^\ell$, in this case, also.

Case 3: Two of k, ℓ, m are 0. Without loss of generality $\ell = m = 0$, leaving $k = 23$.
Then $x^{23} = x^3 x^{16} x^4$ with

$$x \mid x^3,\ y \mid z^4 \mid (x^4)^4 = x^{16},\ z \mid x^4,$$

and hence $xyz \mid x^{23}$.

Therefore, xyz divides $(x+y+z)^{23}$.

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4. Squared circle?

Is it possible to place 2023 consecutive natural numbers around a circle so that the product of each adjacent pair is a perfect square?

Solution. The answer is: No.

Let *pop2* be the *parity of the power of 2* in the prime decomposition of a number, e.g. since $24 = 2^3 \cdot 3$, the *pop2* of 24 is *odd*, and for $12 = 2^2 \cdot 3$, $60 = 3 \cdot 2^2 \cdot 5$ and any odd number, the *pop2* is *even*. Observe that if $ab = c^2$ then the *pop2* of c^2 is *even*, and necessarily the *pop2* of a and b are the same (i.e. both *odd* or both *even*).

Suppose, for a contradiction ($\lightning$), it is possible to place the 2023 consecutive numbers such that the product of each adjacent pair is a square.

Start with an odd number x among the consecutive numbers. Then *pop2* of x is even. So necessarily, the *pop2* of the neighbour of x (to right, say) must be even.

Continuing in this way we see every one of the consecutive numbers must have even *pop2*.

But there are numbers among the consecutive numbers that are 2 *modulo* 4, whose *pop2* is odd$\lightning$.

So, in fact, it's not possible for 2023 consecutive numbers to be arranged around a circle, such that the product of each neighbouring pair is a square.

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5. Nonnegatively polynomial

Suppose $p(x)$ is a polynomial over $\mathbb{R}$ such that

$$p(x) - p'(x) - p''(x) + p'''(x) \geqslant 0, \text{ for all } x \in \mathbb{R}.$$

Prove $p(x) \geqslant 0$ for all $x \in \mathbb{R}$.

Solution. First we prove a lemma.

Lemma. *If for some constant $k \in \mathbb{R}$, $p(x) + kp'(x) \geqslant 0\ \forall x \in \mathbb{R}$, then $p(x) \geqslant 0\ \forall x \in \mathbb{R}$.*

Proof. Assume for some $k \in \mathbb{R}$,

$$p(x) + kp'(x) \geqslant 0, \tag{$*$}$$

for all $x \in \mathbb{R}$.

Since the degree of $kp'(x)$ is one less than the degree of $p(x)$, the behaviour of $p(x) + kp'(x)$ is dominated by the leading term of $p(x)$ as $x \to \infty$ and as $x \to -\infty$. Thus, for $(*)$ to be satisfied, for all real x, it is necessary that $p(x)$ be of even degree and have positive leading coefficient. Hence, $p(x)$ has a global minimum a, and since polynomials are smooth, $p'(a) = 0$.

Now, since $(*)$ is satisfied for all $x \in \mathbb{R}$, in particular, it is satisfied at a, i.e.

$$\begin{aligned} 0 &\leqslant p(a) + kp'(a) \\ &= p(a). \end{aligned}$$

But $p(a)$ is the global minimum of $p(x)$. Hence, for all $x \in \mathbb{R}$,

$$\begin{aligned} p(x) &\geqslant p(a) \\ &\geqslant 0, \end{aligned}$$

and hence the result follows. □

Applying the lemma to $q(x) = p(x) - 2p'(x) + p''(x)$ with $k = 1$, we have $q(x) \geqslant 0 \ \forall x \in \mathbb{R}$, since

$$\begin{aligned} q(x) + q'(x) &= \big(p(x) - 2p'(x) + p''(x)\big) + \big(p'(x) - 2p''(x) + p'''(x)\big) \\ &= p(x) - p'(x) - p''(x) + p'''(x) \\ &\geqslant 0 \text{ (given)}. \end{aligned}$$

Now, applying the lemma again, to $p(x) - p'(x)$ with $k = -1$, we have $p(x) - p'(x) \geqslant 0 \ \forall x \in \mathbb{R}$, since

$$\begin{aligned} \big(p(x) - p'(x)\big) - \big(p(x) - p'(x)\big)' &= p(x) - 2p'(x) + p''(x) \\ &\geqslant 0. \end{aligned}$$

Applying the lemma a third time to $p(x)$ with $k = -1$, finally delivers the required property of $p(x)$.