Blakers Mathematics Competition 2024

UWA Mathematics Union

Open to first to third year students of any Western Australian university, with prizes sponsored by the UWA Mathematics Union.

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers), and a lightning bolt ($\lightning$) signals a contradiction.

2024 Problems with Solutions

1. Radical annual empowered

Does there exist $N \in \mathbb{N}$ such that

$$(\sqrt{2024} - \sqrt{2023})^{2022} = \sqrt{N} - \sqrt{N-1}?$$

Solution. Consider the more general problem

$$(\sqrt{x+1} - \sqrt{x})^n = \sqrt{N} - \sqrt{N-1} \qquad (*)$$

where $x, n \in \mathbb{N}$ and $2 \,|\, n$, i.e. $n = 2m$, for some $m \in \mathbb{N}$. Then

$$\begin{aligned}
(\sqrt{x+1} - \sqrt{x})^n &= \sum_{r=0}^{n} \binom{n}{r} \left(\sqrt{x+1}\right)^{n-r} (-1)^r \left(\sqrt{x}\right)^r \\
&= \sum_{\substack{s=0 \\ r=2s \\ r \text{ even}}}^{m} \binom{2m}{2s} (x+1)^{m-s} x^s - \sum_{\substack{s=1 \\ r=2s-1 \\ r \text{ odd}}}^{m} \binom{2m}{2s-1} (x+1)^{m-s} x^s \cdot \frac{\sqrt{x+1}}{\sqrt{x}} \\
&= S_1 - S_2,
\end{aligned}$$

where S_1 is the first sum (over even r) and $-S_2$ is the remaining sum (over odd r). Similarly,

$$(\sqrt{x+1} + \sqrt{x})^n = S_1 + S_2$$

$$\begin{aligned}
\therefore\ (\sqrt{x+1} + \sqrt{x})^n (\sqrt{x+1} - \sqrt{x})^n &= (S_1 + S_2)(S_1 - S_2) \\
= (x + 1 - x)^n \qquad &= S_1^2 - S_2^2 \\
= 1 \qquad &
\end{aligned}$$

We see from how S_1 is defined, that $S_1 \in \mathbb{N}$. Hence, $S_1^2, S_2^2 = S_1^2 - 1 \in \mathbb{N}$. So with $N = S_1^2$, we have

$$\begin{aligned}
(\sqrt{x+1} - \sqrt{x})^n &= S_1 - S_2 \\
&= \sqrt{N} - \sqrt{N-1}.
\end{aligned}$$

So, yes, there does exist an $N \in \mathbb{N}$ satisfying $(*)$. For the given problem, $x = 2023, n = 2022$, and $m = 1011$, so that

$$N = S_1^2 = \left(\sum_{s=0}^{1011} \binom{2022}{2s} 2024^{1011-s} 2023^s \right)^2.$$

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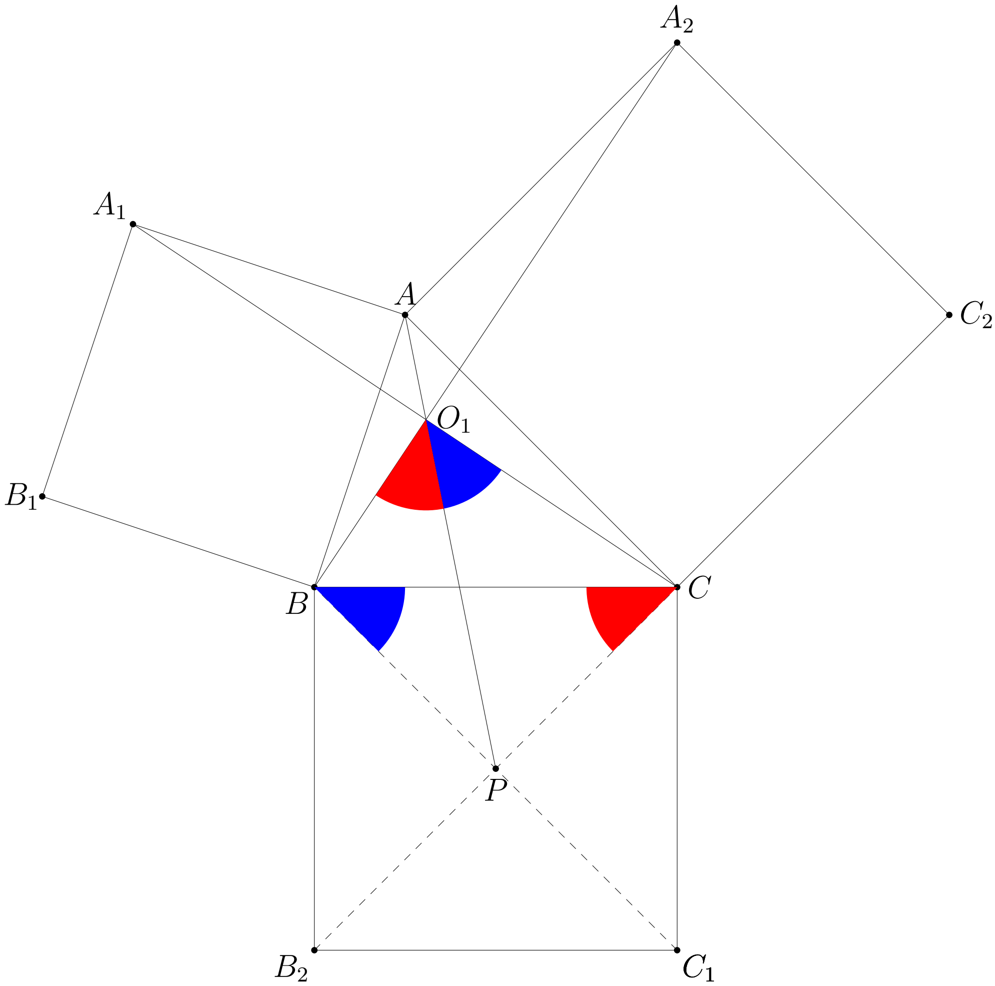
2. Squared triangle

On sides AB, BC and CA of $\triangle ABC$, construct squares ABB_1A_1, BCC_1B_2, CAA_2C_2, respectively.
Let P be the centre of square BCC_1B_2.
Prove lines A_1C, A_2B an AP concur.
Note. Lines *concur* if they pass through a common point.

Solution. Let $\rho(X, \theta, k)$ be the spiral symmetry that rotates about point X through an angle θ while dilating by ratio k. We use two spiral symmetries, below.



Let O_1 and O_2 to be the points where AP meet A_1C and A_2B, respectively. Our strategy is show O_1 and O_2 are the same point.

$$\begin{aligned} \rho(B, -45^\circ, 1/\sqrt{2}) : A_1 &\mapsto A \\ C &\mapsto P \\ A_1C &\mapsto AP \\ O_1 &= A_1C \cap AP \\ \text{Then } 45^\circ &= \angle A_1C, AP \\ &= \angle CO_1P \\ \text{Also } 45^\circ &= \angle CBP \\ \therefore\ PBO_1C &\text{ is cyclic.} \end{aligned}$$

So we have that O_1 is the point other than P where AP meets circumcircle(CBP). Similarly,

$$\begin{aligned}
\rho(C, 45^\circ, 1/\sqrt{2}) : A_2 &\mapsto A \\
B &\mapsto P \\
A_2B &\mapsto AP \\
O_2 &= A_2B \cap AP \\
\text{Then } 45^\circ &= \angle A_2B, AP \\
&= \angle BO_2P \\
\text{Also } 45^\circ &= \angle BCP \\
\therefore\ PBO_2C &\text{ is cyclic.}
\end{aligned}$$

So we have that O_2 is the point other than P where AP meets circumcircle(CBP), which is to say $O_1 = O_2$, or equivalently that lines A_1C, A_2B and AP concur.

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3. Fractional sum

Show that

$$\frac{2023}{2} - \frac{2022}{3} + \frac{2021}{4} + \cdots - \frac{2}{2023} + \frac{1}{2024} = \frac{1}{1013} + \frac{3}{1014} + \cdots + \frac{2023}{2024} \tag{$*$}$$

Solution. Let

$$S(n) = \sum_{k=2}^{2n} \frac{(-1)^k}{k} + \sum_{k=n+1}^{2n} \frac{1}{k}.$$

First we prove $S(n) = 1$, by induction.

(base case) We prove $S(1) = 1$.

$$\begin{aligned}
S(1) &= \sum_{k=2}^{2} \frac{(-1)^k}{k} + \sum_{k=2}^{2} \frac{1}{k} \\
&= \tfrac{1}{2} + \tfrac{1}{2} \\
&= 1.
\end{aligned}$$

(inductive step) We prove: $S(n) = 1 \implies S(n+1) = 1$, for general n, by showing $S(n+1) - S(n) = 0$:

$$\begin{aligned}
S(n+1) - S(n) &= \sum_{k=2}^{2(n+1)} \frac{(-1)^k}{k} + \sum_{k=(n+1)+1}^{2(n+1)} \frac{1}{k} \\
&\quad - \Big(\sum_{k=2}^{2n} \frac{(-1)^k}{k} + \sum_{k=n+1}^{2n} \frac{1}{k}\Big) \\
&= -\frac{1}{2n+1} + \frac{1}{2n+2} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\
&= \frac{2}{2n+2} - \frac{1}{n+1} \\
&= 0.
\end{aligned}$$

So the induction is complete and hence $S(n) = 1$ for all $n \in \mathbb{N}$.

Now,

$$\begin{aligned}
\text{RHS}(*) &= \frac{1}{1013} + \frac{3}{1014} + \cdots + \frac{2023}{2024} \\
&= \Big(\frac{1}{1013} - 2\Big) + \Big(\frac{3}{1014} - 2\Big) + \cdots + \Big(\frac{2023}{2024} - 2\Big) + 2(2024 - 1013 + 1) \\
&= 2024 - 2025\Big(\frac{1}{1013} + \frac{1}{1014} + \cdots + \frac{1}{2024}\Big) \\
\text{LHS}(*) &= \frac{2023}{2} - \frac{2022}{3} + \frac{2021}{4} + \cdots - \frac{2}{2023} + \frac{1}{2024} \\
&= \Big(\frac{2023}{2} + 1\Big) - \Big(\frac{2022}{3} + 1\Big) + \cdots + \Big(\frac{1}{2024} + 1\Big) - 1 \\
&= 2025\Big(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2024}\Big) - 1 \\
\therefore\ \text{RHS}(*) - \text{LHS}(*) &= 2025 - 2025 \cdot S(2024) \\
&= 0,
\end{aligned}$$

and hence $(*)$ holds.

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4. Radically signed

Somewhat informally, let $a_1 = 2 \pm \sqrt{2}$, $a_{n+1} = 2 \pm \sqrt{a_n}$, $n \geqslant 1$.
Let $A(n)$ be the set of all expressions a_n, e.g.

$$A(2) = \Big\{2 + \sqrt{2 + \sqrt{2}}, 2 + \sqrt{2 - \sqrt{2}}, 2 - \sqrt{2 + \sqrt{2}}, 2 - \sqrt{2 - \sqrt{2}}\Big\}.$$

(a) Prove that all the elements of $A(n)$ are real.

(b) Compute the product

$$\prod_{a \in A(n)} a.$$

(c) If $A(24)$ is sorted into ascending order, what position is the element whose signs in order are

$$- - + + + + + + + + - - - - + + + + + - - + + - + \,.$$

Solution.

(a) In order to prove elements of $A(n)$ are real, let

$$P(n) : A(n) \subset (0, 4).$$

We will prove $P(n)$ holds for all $n \in \mathbb{N}$ by induction, noting that by $(0, 4)$ we mean the open real interval (from 0 to 4).

(base case) We prove $P(1)$.

$$A(1) = \{2 - \sqrt{2}, 2 + \sqrt{2}\}$$

$$\begin{aligned} \text{Since, } 0 < \sqrt{2} < 2,& \\ 0 < 2 - \sqrt{2}& \\ < 2 + \sqrt{2}& \\ < 4& \\ \therefore\ A(1) \subset (0, 4)& \\ \therefore\ P(1) \text{ holds.}& \end{aligned}$$

(inductive step) We prove: $P(n) \implies P(n+1)$.
Assume $P(n)$ holds, i.e. for each $a_n \in A(n), a_n \in (0, 4)$.
Then

$$\begin{aligned} \text{each } \sqrt{a_n} &\in (0, 2) \\ \therefore\ \text{each } a_{n+1} &= 2 \pm \sqrt{a_n} \\ &\in (2 - 2, 2 + 2) = (0, 4) \\ \therefore\ P(n+1) &\text{ holds, if } P(n) \text{ holds.} \end{aligned}$$

(conclusion) So, the induction is complete.
i.e. $P(n)$ holds for all $n \in \mathbb{N}$.
i.e. $A(n) \subset (0, 4)$ for all $n \in \mathbb{N}$.
In particular, all the elements of each $A(n)$ are real.

(b) Let $p_n(x)$ be the monic polynomial whose zeros are all the a_n:

$$\begin{aligned} p_n(x) &= \prod_{a_n \in A(n)} (x - a_n) \\ &= \prod_{a_{n-1} \in A(n-1)} \big((x - 2 + \sqrt{a_{n-1}})(x - 2 - \sqrt{a_{n-1}})\big) \\ &= \prod_{a_{n-1} \in A(n-1)} \big((x-2)^2 - a_{n-1}\big) \\ &= \prod_{a_{n-2} \in A(n-2)} \big(((x-2)^2 - 2)^2 - a_{n-2}\big) \\ &\ \vdots \\ &= \prod_{a_{n-k} \in A(n-k)} \big((\cdots((x-2)^2 - 2)^2 - \cdots - 2)^2 - a_{n-k}\big),\ (k+1) \text{ pairs of brackets} \\ &\ \vdots \\ &= \prod_{a_1 \in A(1)} \big((\cdots((x-2)^2 - 2)^2 - \cdots - 2)^2 - a_1\big),\ n \text{ pairs of brackets} \\ &= \big((\cdots((x-2)^2 - 2)^2 - \cdots - 2)^2 - 2\big),\ n+1 \text{ pairs of brackets.} \end{aligned}$$

Now $p_n(0)$ is the constant term of $p_n(x)$ which is the product of the zeros of $p_n(x)$ and observe that

$$(0-2)^2 - 2 = 2,$$

so that,

$$\begin{aligned} \prod_{a \in A(n)} a &= \big((\cdots((0-2)^2 - 2)^2 - \cdots - 2)^2 - 2\big) \\ &= \big((\cdots(2^2 - 2)^2 - \cdots - 2)^2 - 2\big) \\ &\ \vdots \\ &= 2. \end{aligned}$$

(c) Since $A(1)$ consists of 2 elements, and $A(n)$ has twice as many elements as $A(n-1)$, it follows (by induction) that $A(n)$ has 2^n elements.
Let us identify each element of $A(n), A(n-1), \dots, A(1)$ with its string of signs.
Our task is to find the ordinal of $--++++++++----++++--++-+$ in $A(24)$, a number in $\{1, 2, 3, \dots, 2^{24} = 16\,777\,216\}$.
For $s \in A(24)$ (where s is identified by its string of signs), let $\operatorname{ord}(s)$ be the ordinal of s in $A(24)$.
Subtracting 1 from $\operatorname{ord}(s)$, and writing in binary, zero-filling on the left if necessary, associates a 24-digit binary string with each s.
First observe that if s is the "all +" string, then s is the largest element of $A(24)$; so $\operatorname{ord}(s) - 1 = 2^{24} - 1$, a string of all 1s.
Next observe that the string s starting with $-$ and followed by 23 +s is the least element of $A(24)$, so that $\operatorname{ord}(s) - 1$ is a string of all 0s.
Finally, observe that if we have written all the elements of $A(n-1)$ in increasing order, then:

> inserting a $+$ at the beginning of each of these elements, the resulting elements of $A(n)$ are all in the correct order, but
> inserting a $-$ at the beginning of each of the ordered $A(n-1)$ elements, results in a sequence of $A(n)$ elements that are in reverse order.

Moreover, all $A(n)$ elements beginning with a $+$ are larger than all the elements beginning with a $-$.
Applying this idea recursively, we see that we can construct $\operatorname{ord}(s) - 1$ in binary, by reading the signs in s from the left, and

1. Start with $d = 1$.
2. While the sign in s is $+$, output d.
3. While the sign in s is $-$, toggle d (from 0 to 1, or from 1 to 0), and output d.

Thus,

$$\begin{aligned}
&\operatorname{ord}(--++++++++----++++--++-+) \\
&\quad = 0\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,1\,0\,1\,1\,1\,1\,1\,0\,1\,1\,1\,0\,0_2 + 1 \\
&\quad = 2^{23} - 1 - 2^{13} - 2^{11} - 2^5 - 2^1 - 2^0 + 1 \\
&\quad = 2^{23} - 2^{13} - 2^{11} - 2^5 - 3 \\
&\quad = 8\,378\,333.
\end{aligned}$$

So, $--++++++++----++++--++-+$ is the $8\,378\,333^{\text{th}}$ element of $A(24)$.

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5. Integer and fractionally parted

For $x, y \in \mathbb{N}$, let $\overline{x.y}$ be the number whose integer part is composed of the decimal digits of x, and whose fractional part is composed of the decimal digits of y, e.g. if $x = 124$ and $y = 816$ then $\overline{x.y} = 124.816$.
Find all solutions of

$$\frac{a}{b} = \overline{b.a},$$

where a, b are coprime.

Solution. We have $a, b \in \mathbb{N}$, with a, b coprime.
We first note that neither a nor b can be 1, since

$$\begin{aligned} b = 1 &\implies a = 0 \\ &\implies \overline{b.a} = 0 \ ↯; \text{ and} \\ a = 1, b > 1 &\implies a/b < 1, \overline{b.a} > 1 \ ↯. \end{aligned}$$

Let $n \in \mathbb{N}$ be such that $10^{n-1} \leqslant a < 10^n$. Then

$$\begin{aligned} \frac{a}{b} &= \overline{b.a} \\ &= b + a \cdot 10^{-n} \\ \therefore\ a &= b^2 + ab \cdot 10^{-n} \\ \therefore\ a - b^2 &= ab \cdot 10^{-n} \\ \therefore\ 10^n &= \frac{ab}{a - b^2} \end{aligned} \tag{1}$$

Suppose, for a contradiction, ab and $a - b^2$ are not coprime.
Then $ab, a - b^2$ have a common prime divisor p. Hence,

$$\begin{aligned} p &\,\big|\, \big(a^2(a-b^2) + (ab)^2\big) = a^3 \text{ and} \\ p &\,\big|\, \big(ab - b(a-b^2)\big) = b^3 \\ \therefore\ p &\,\big|\, a \text{ and } p \,\big|\, b \\ \therefore\ &a, b \text{ are not coprime} ↯ \\ \therefore\ &\text{In fact, } ab, a - b^2 \text{ are coprime} \\ \therefore\ &\text{by (1): } a - b^2 = 1, ab = 10^n \\ \therefore\ a &= b^2 + 1 \text{ and } \{a, b\} = \{2^n, 5^n\}, \text{ since } 1 \neq a, b \text{ coprime} \\ &> b \\ \therefore\ &(a, b) = (5^n, 2^n). \end{aligned}$$

Now, consider cases according to n:

Case 1: $n \geqslant 2$. Then

$$\begin{aligned} a &= 5^n \\ &= (1+4)^n \\ &\geqslant 1 + n4^{n-1} + 4^n \\ &\geqslant 1 + 2 \cdot 4 + 4^n \\ &> 1 + 4^n \\ &= 1 + (2^n)^2 \\ &= 1 + b^2 \ \lightning \end{aligned}$$

which leaves ...

Case 2: $n = 1$, whence,

$$\begin{aligned} a &= 5 \\ &= 1 + 2^2 \\ &= 1 + b^2. \end{aligned}$$

Thus, $(a, b) = (5, 2)$ which gives

$$\frac{a}{b} = \tfrac{5}{2} = 2.5 = \overline{b.a},$$

is the only solution.