Blakers Mathematics Competition 2026

Open to first to third year students of any Western Australian university, with prizes sponsored by the UWA Mathematics Union.

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers), a lightning bolt (↯) signals a contradiction, and serial exponentiation is *right-associative*, i.e.

$$a^{b^c} = a^{(b^c)}.$$

2025 Problems with Solutions

1. Legion amusement

Legion's numbers of the first, second and third kinds are

$$L_1 = 666^{666},$$
$$L_2 = 666!^{666!},$$
$$L_3 = 666^{666^{666}},$$

respectively. Also, denote the sum of the digits of N, by $S(N)$.

(a) Find the sum of the last 666 666 digits of L_2.

(b) Which is larger $S(L_2)$ or 10^{2000}?

(c) Which is larger L_2 or L_3?

(d) Find $S(S(S(S(L_3))))$?

Solution.

(a) The p**-adic valuation** of N, denoted by $\nu_p(N)$, is the highest power of the prime p that divides N.

$$\begin{aligned}\nu_2(666!) > \nu_5(666!) &= \lfloor 666/5 \rfloor + \lfloor 666/25 \rfloor + \lfloor 666/125 \rfloor + \lfloor 666/625 \rfloor \\ &= 133 + 26 + 5 + 1 \\ &= 165.\end{aligned}$$

Hence, 10^{165} is the highest power of 10 dividing 666!, and,

$$165 \cdot 666! > 165 \cdot 666 \cdot 665 = 73\,076\,850 > 666\,666.$$

So, $10^{666\,666} \mid L_2$, i.e. the last 666 666 digits of L_2 are all 0s.
Therefore, the sum of the last 666 666 digits of L_2 is 0.

(b) We have,

$$\begin{aligned}
666! &< 666^{666} \\
&< \left(\frac{2000}{3}\right)^{666} \\
&= (\tfrac{2}{3})^{666} \cdot 1000^{666} \\
&= \left((\tfrac{2}{3})^{6}\right)^{111} \cdot 1000^{666} \\
&= (\tfrac{64}{729})^{111} \cdot 1000^{666} \\
&< (\tfrac{1}{10})^{111} \cdot 10^{1998} \\
&= 10^{1887} \\
\therefore\ L_2 = 666!^{666!} &< (10^{1887})^{10^{1887}} \\
&= 10^{1887\cdot 10^{1887}} \\
\therefore\ S(L_2) &\leqslant 9 \cdot 1887 \cdot 10^{1887}, \text{ since of numbers } M < 10^N, \text{ the number with largest digit sum is the number with } N \text{ digits that are all 9s} \\
&< 10^{2000}.
\end{aligned}$$

So 10^{2000} is the larger of $S(L_2)$ and 10^{2000}.

(c) We have,

$$\begin{aligned}
666^{665} &= 666 \cdot 666 \cdots 666 \\
&> 666 \cdot 665 \cdots 2 \cdot 1 \\
&= 666! \\
\therefore\ L_3 = 666^{666^{666}} &= 666^{(666^{665} \cdot 666)} \\
&= \left(666^{666^{665}}\right)^{666} \\
&> \left(666^{666!}\right)^{666} \\
&= \left(666^{666}\right)^{666!} \\
&> 666!^{666!} = L_2.
\end{aligned}$$

So L_3 is larger than L_2.

(d) Firstly, note that, since generally, for $N \in \mathbb{N}$,

$$N \equiv S(N) \pmod 9,$$

we have,

$$\begin{aligned}
&L_3 \equiv S(L_3) \equiv S(S(L_3)) \equiv S(S(S(L_3))) \equiv S(S(S(S(L_3)))) \pmod 9, \text{ and} \\
&L_3 = 666^{666^{666}} \\
&\quad \equiv (6+6+6)^{666^{666}} \pmod 9 \\
&\quad \equiv 0 \pmod 9 \\
\therefore\ &S(S(S(S(L_3)))) \equiv 0 \pmod 9
\end{aligned}$$

Now we bound $S(S(S(S(L_3))))$.

The general idea is that given an N that is at most B, in bounding $S(N)$ we look for the number less than B with the largest digit sum, which is typically a number with a similar number of digits to B, but whose digits are almost all 9s.
For example, for $B = 16993$, the number $M \leqslant B$ with the largest digit sum is $M = 9999$.
Thus,

$$\begin{aligned}
L_3 &= 666^{666^{666}} \\
&< 1000^{666^{666}} \\
&< 1000^{10^{1887}}, \text{ from a partial calculation in (b)} \\
&= 10^{3\cdot 10^{1887}} \\
\therefore\ S(L_3) &\leqslant 9\cdot 3\cdot 10^{1887}, \text{ for the same reason we gave when bounding } S(L_2) \text{ in (b)} \\
&= 27\cdot 10^{1887} \\
\therefore\ S(S(L_3)) &\leqslant S(199\dots 9), \text{ where there are 1888 `9's} \\
&= 1 + 1888\cdot 9 \\
&= 16993 \\
\therefore\ S(S(S(L_3))) &\leqslant S(9999) \\
&= 36 \\
\therefore\ S(S(S(S(L_3)))) &\leqslant S(29) \\
&= 11.
\end{aligned}$$

So $0 < S(S(S(S(L_3)))) < 11$ and $S(S(S(S(L_3)))) \equiv 0 \pmod 9$.
The only possibility is: $S(S(S(S(L_3)))) = 9$.

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2. Fourth empowerment

Let $S = \{1, 4, 9, \dots, k^2, \dots, 2025\}$ and let T be the subset of S such that no pair of distinct elements of T has a product that is a perfect fourth power.

(a) Find the maximum cardinality of T.

(b) Find the number of such sets T of maximum cardinality.

Solution. Equivalently, let $S' = \{1, 2, 3, \dots, 45\}$ and let T' be the subset of S' such that no pair of distinct elements of T' has a product that is a perfect square.
For convenience define Property P to be the property that no pair of distinct elements (of T') has a product that is a perfect square, and Property Q to be the corresponding property of T with "square" replaced by "fourth power".
Define an equivalence relation $\sim$ on S' by,

$$x \sim y \iff xy = t^2 \text{ for some } t \in \mathbb{N}.$$

Checking,

$$x \sim x, \text{ since } x^2 \text{ is a perfect square.} \qquad \text{So } \sim \text{ is } \textit{reflexive.}$$

$$\begin{aligned}
x \sim y &\implies xy = t^2 \text{ for some } t \in \mathbb{N} \\
&\implies yx = t^2 \text{ for some } t \in \mathbb{N} \\
&\implies y \sim x \qquad \text{So } \sim \text{ is } \textit{symmetric.}
\end{aligned}$$

$$\begin{aligned}
x \sim y, y \sim z &\implies xy = t^2, yz = u^2 \text{ for some } t, u \in \mathbb{N} \\
&\implies xz = (tu/y)^2, \text{ a square in } \mathbb{N} \\
&\implies x \sim z \qquad \text{So } \sim \text{ is } \textit{transitive.}
\end{aligned}$$

So $\sim$ is indeed an equivalence relation.
Observe that $x \sim y$ if x and y have equal square-free component. Thus the equivalence classes on S' are:

$$\begin{array}{ll}
\{1,4,9,16,25,36\}, & \{23\}, \\
\{2,8,18,32\}, & \{26\}, \\
\{3,12,27\}, & \{29\}, \\
\{5,20,45\}, & \{30\}, \\
\{6,24\}, & \{31\}, \\
\{7,28\}, & \{33\}, \\
\{10,40\}, & \{34\}, \\
\{11,44\}, & \{35\}, \\
\{13\}, & \{37\}, \\
\{14\}, & \{38\}, \\
\{15\}, & \{39\}, \\
\{17\}, & \{41\}, \\
\{19\}, & \{42\}, \\
\{21\}, & \{43\}. \\
\{22\}, &
\end{array}$$

These are generated by:

first observing that all the perfect squares in S' form a class.
Then each subsequent class is generated by identifying the smallest element of S' not yet seen, and multiplying that element by squares until too large to be in S'.

(a) Taking at most one element from each equivalence class gives a subset T' of S' with property P.
Since there are 29 equivalence classes, the maximum cardinality of T' is 29.
(By the Pigeon-Hole Principle, a subset of S' with 30 or more elements necessarily has 2 elements from the same equivalence class, so that the subset fails to have property P.)
Squaring each element of a T' with property P gives a subset T of S with property Q.
So the maximum cardinality of a set T with property Q is also 29.

(b) The

$$\begin{aligned}
&\text{“number of maximal sets } T \text{ with property } Q\text{”} \\
&= \text{“number of maximal sets } T' \text{ with property } P\text{”} \\
&= \text{“number of ways we can choose one element from each equivalence class”} \\
&= \text{“product of the cardinalities of the equivalent classes”} \\
&= 6 \cdot 4 \cdot 3^2 \cdot 2^4 \cdot 1^{21} \\
&= 2 \cdot 12^3 \\
&= 3456.
\end{aligned}$$

3. Hexadecimal tetration

What is the last hexadecimal digit of

$$B^{B^{B^{\cdot^{\cdot^{\cdot^{B}}}}}},$$

where there are B Bs?

Note. Hexadecimal is base sixteen, which has digits $0, 1, 2, \dots, 9, A, B, C, D, E, F$. You may find it convenient to refer to such an expression as a *tower* of so many Bs, or to a *tetration*. The n^{th} **tetration** of a, written ${}^{n}a$, means $a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}$ (where there are n copies of a).

Solution. First we note that B is the decimal number 11.
Also, finding the last hexadecimal digit is equivalent to performing a "mod 16" operation.
By Euler's Theorem, since 11 is coprime to 16,

$$\begin{aligned}
11^{\varphi(16)} &\equiv 1 \pmod{16}. \\
\therefore\ {}^{B}B \mod 16 &= B^{\left({}^{B-1}B \mod \varphi(16)\right)} \mod 16, \text{ and} \\
\varphi(16) &= (1 - \tfrac{1}{2}) \cdot 16 \\
&= 8.
\end{aligned}$$

Similarly,

$$\begin{aligned}
{}^{B-1}B \mod \varphi(16) &= {}^{B-1}B \mod 8 \\
&= B^{\left({}^{B-2}B \mod \varphi(8)\right)} \mod 8 \\
&= B^{\left({}^{B-2}B \mod 4\right)} \mod 8 \\
{}^{B-2}B \mod 4 &= B^{\left({}^{B-3}B \mod 2\right)} \mod 4 \\
&= B^{1} \mod 4, \text{ since } B \text{ is odd.} \\
&= 3 \\
\therefore\ {}^{B-1}B \mod 8 &= B^{3} \mod 8 \\
&= 3^{3} \mod 8 \\
&= 3 \\
\therefore\ {}^{B}B \mod 16 &= B^{3} \mod 16 \\
&= (-5)^{3} \mod 16 \\
&= 25 \cdot (-5) \mod 16 \\
&= 9 \cdot (-5) \mod 16 \\
&= -45 \mod 16 \\
&= 3.
\end{aligned}$$

So, the last hexadecimal digit of ${}^{B}B$ is 3.

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4. Powerfully functional

Let function $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x^3 + y^3) = (x + y)\big(f(x)^2 - f(x)f(y) + f(y)^2\big) \qquad (*)$$

for all $x, y \in \mathbb{R}$.
Prove $f(2025x) = 2025f(x)$.

Solution. Putting $x = y = 0$ in $(*)$, gives $f(0) = 0$.
Setting $y = 0$ in $(*)$, gives for all $x \in \mathbb{R}$,

$$f(x^3) = xf(x)^2 \qquad (1)$$

or equivalently, for all $x \in \mathbb{R}$,

$$f(x) = \sqrt[3]{x}f(\sqrt[3]{x})^2, \qquad (2)$$

Observe that x and $f(x)$ must have the same sign.
Let $S = \{k \in \mathbb{R} : f(kx) = kf(x)\ \forall\, x \in \mathbb{R}\}$.
Trivially, we have $1 \in S$.
Take $k \in S$. Then

$$\begin{aligned} kxf(x)^2 &= kf(x^3), \text{ by (1)} \\ &= f(kx^3), \text{ since } k \in S \\ &= f\big((\sqrt[3]{k}\,x)^3\big) \\ &= \sqrt[3]{k}\,xf(\sqrt[3]{k}\,x)^2, \text{ by (2)} \end{aligned}$$

or equivalently,

$$\big(\sqrt[3]{k}\,f(x)\big)^2 = f(\sqrt[3]{k}\,x)^2,$$

which, since x and $f(x)$ have the same sign, is equivalent to,

$$\sqrt[3]{k}\,f(x) = f(\sqrt[3]{k}\,x).$$

So, we have $k \in S \implies \sqrt[3]{k} \in S$.
Now suppose $h, k \in S$. Then,

$$\begin{aligned} f\big((h+k)x\big) &= f(hx + kx) \\ &= f\big((\sqrt[3]{hx})^3 + (\sqrt[3]{kx})^3\big) \\ &= (\sqrt[3]{hx} + \sqrt[3]{kx})\big(f(\sqrt[3]{hx})^2 - f(\sqrt[3]{hx})f(\sqrt[3]{kx}) + f(\sqrt[3]{kx})^2\big), \text{ by } (*) \\ &= (\sqrt[3]{h} + \sqrt[3]{k})\sqrt[3]{x}\big((\sqrt[3]{h})^2 - \sqrt[3]{h}\,\sqrt[3]{k} + (\sqrt[3]{k})^2\big)f(\sqrt[3]{x})^2, \text{ since } h, k \in S \implies \sqrt[3]{h}, \sqrt[3]{k} \in S \\ &= (h+k)f(x), \text{ by (2)} \end{aligned}$$

So $h, k \in S \implies h + k \in S$.
Hence, since $1 \in S$, we have $1 + 1 = 2 \in S$, and by induction $n \in S$ for all $n \in \mathbb{N}$.
In particular, $2025 \in S$.
Hence, $f(2025x) = 2025f(x)$.

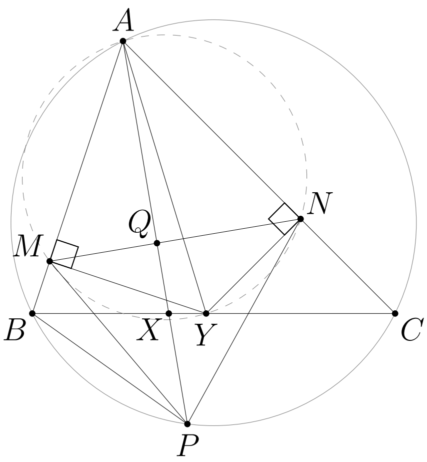
5. Common area

Suppose points X and Y lie in that order on side BC of acute triangle ABC, such that $\angle BAX = \angle CAY$.
Points M, N lie on AB, AC, respectively, such that $YM \perp AB$ and $YN \perp AC$.
Line AX produced intersects the circumcircle of ABC again at P.
Prove the areas of triangle ABC and quadrilateral $AMPN$ are equal.

Solution. At a key step, we use:

Theorem (Extended Sine Rule)**.** *In $\triangle ABC$ with circumradius R,*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$



Let $Q = MN \cap AP$

$\angle AMN = 90^\circ = \angle ANY$

$\therefore$ $AMYN$ is cyclic, (opposite angles at M, N are supplementary)

$\therefore$ $\dfrac{MN}{\sin A} = AY$, by the Extended Sine Rule, since in $\triangle AMN$,

$A = \angle MAN = \angle BAC$,

$a = MN$,

$2R = AY = \text{circumdiameter}(AMN)$

$\therefore$ $MN = AY \cdot \sin A$

Also,

$$\begin{aligned}
\angle AMN &= \angle AYN, \text{ (angles on arc } AN \text{ in cyclic } AMYN)\\
\therefore\ \angle AQM &= 180^\circ - (\angle AMQ + \angle MAQ)\\
&= 180^\circ - (\angle AMN + \angle BAX), \text{ (same angles)}\\
&= 180^\circ - (\angle AYN + \angle CAY)\\
&= \angle ANY\\
&= 90^\circ\\
\therefore\ |AMPN| &= \tfrac{1}{2} AP \cdot MN\\
\angle APB &= \angle ACB \text{ (angles on arc } AB \text{ in cyclic } ACPB)\\
&= \angle ACY, \text{ (same angle)}\\
\angle BAP &= \angle BAX, \text{ (same angle)}\\
&= \angle CAY\\
&= \angle YAC, \text{ (same angle)}\\
\therefore\ \triangle BAP &\sim \triangle YAC, \text{ by AA Rule}\\
\therefore\ \frac{AB}{AP} &= \frac{AY}{AC}\\
\therefore\ AB \cdot AC &= AP \cdot AY\\
\therefore\ |ABC| &= \tfrac{1}{2} bc \sin A\\
&= \tfrac{1}{2} \cdot AB \cdot AC \sin A\\
&= \tfrac{1}{2} \cdot AP \cdot AY \sin A\\
&= \tfrac{1}{2} \cdot AP \cdot MN\\
&= |AMPN|.
\end{aligned}$$

Thus, the areas of triangle ABC and quadrilateral $AMPN$ are equal.